Question

The waiting list for a particular operation at a hospital consists of j people. The consultant performs at most one operation a day and there is a probability $Q(>\frac{1}{2})$ that he will operate on a day. The probability of one new patient being added to the waiting list on a day is P, and the probability of no new patient being added is $(1-P)$. A patient will not have an operation on the same day he is added to the waiting list. When there are $a(>j)$ people on the waiting list the computer goes on strike and all new patients are directed to a different hospital. Show that the length of the waiting list can be described as a simple random walk with two barriers, stating any necessary assumptions. Find the probability that the length of the waiting list reaches a. If $Q=\frac{2}{3}$ and $P=\frac{1}{3}$ find the expected number of days until a strike occurs.

Answer

Let X_n = length of waiting list at day n

Let Z_n = change in waiting list on day n

Then $X_0=j$ and $X_n=X_{n-1}+Z_n \quad n=1,2,3,\ldots$

Assume that the decision to perform operation is independent of whether a new patient is added.

Then $Z_n=\begin{cases} 1 & \text{with probabiltiy } P(1-Q)=p \\ -1 & \text{with probability } (1-P)Q=q \\ 0 & \text{with probability } (1-P)(1-Q)+PQ=r \end{cases}$

provided $0<X_{n+1}<a$

$$\begin{aligned} Z_n &= 0 \text{ with probability 1when } X_{n-1}=a \\ Z_n &= \left\{\begin{array}{ll} 1 & \text{with probability } P \\ 0) & \text{with probability } 1-P \end{array}\right. \text{ when } X_{n-1}=0 \end{aligned}$$

Assuming that activities on different days for both surgeon and patients are independent, the $Z_n's$ are independent and (X_n) is a simple random walk with an absorbing barrier at a and a reflecting barrier at 0.

Let q_j = probability of absorption at a from a start at j

$$\begin{aligned} \text{Then} \quad q_j &= pq_{j+1}+qq_{j-1}+rq_j \quad j=1,2,\ldots,a-1 \\ q_a &= 1 \\ q_0 &= Pq_1+(1-P)q_0 \text{ i.e. } q_1=q_0 \end{aligned}$$

The general solution of the difference equation is

$$\begin{aligned} q_j &= A\left(\frac{q}{p}\right)^j \qquad q\neq p \\ q_j &= A+Bj \qquad q=p \end{aligned}$$

To find A and B
Case I: $p \neq q$
$q_a = 1$ so $A\left(\frac{q}{p}\right)^a + B = 1$
As $q_1 = q_0$ we get $A + B = A\left(\frac{q}{p}\right) + B \Rightarrow A = 0$ therefore $B = 1$

Case II: $p = q$
$q_a = 1$ so $A + Ba = 1$
As $q_1 = q_0$ we get $A = A + B \Rightarrow B = 0$ therefore $A = 1$
Hence $q_j = 1$ in both cases. Absorption is certain.

Now let E_j be the expected number of days until absorption.

$$\begin{aligned} E_j &= p(1 + E_{j+1}) + q(1 + E_{j-1}) + r(1 + E_j) \qquad j = 1, 2, \ldots, a-1 \\ E_0 &= 0 \\ E_0 &= P(1 + E_1) + (1 - P)(1 + E_0) \quad \text{i.e. } 1 = P(E_0 + E_1) \end{aligned}$$

The general solution of the difference equation is

$$E_j = A + B\left(\frac{q}{p}\right)^j - \frac{j}{p-q} \quad \text{for } p \neq q$$

When $Q = \frac{2}{3}$ and $P = \frac{1}{3}$; then $p = \frac{1}{9}$ and $q = \frac{4}{9}$. Using these values and the boundary conditions, gives

$$B = -2 \quad \text{and} \qquad A = 2 \cdot 4^a - 3a$$

So

$$E_j = 2(4^a - 4^j) + 3(j - a)$$