Question

Suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are two σ-algebras of sets

i) Is $\mathcal{M}_1 \cup \mathcal{M}_2$ necessarily a σ-algebra?

ii) Is $\mathcal{M}_1 \cap \mathcal{M}_2$ necessarily a σ-algebra?

iii) Is $\mathcal{M}_1 - \mathcal{M}_2$ necessarily a σ-algebra?

Answer

i) Let $\mathcal{M}_1$ be the collection of countable subsets of $[0,1]$, together with their complements. Then $\mathcal{M}_1$ is a σ-algebra. Let $\mathcal{M}_2$ be the collection of countable subsets of $[1,2]$ together with their complements. Then $\mathcal{M}_2$ is a σ-algebra, $\mathcal{M}_1 \cup \mathcal{M}_2$ is not a σ-algebra, since it is not closed under unions.

ii) $\mathcal{M}_1 \cap \mathcal{M}_2$ is a σ-algebra.

$E \epsilon \mathcal{M}_1 \cap \mathcal{M}_2 \Rightarrow E \epsilon \mathcal{M}_1 \wedge E \epsilon \mathcal{M}_2 \Rightarrow E^C \epsilon \mathcal{M}_1 \wedge E^C \epsilon \mathcal{M}_2 \Rightarrow E^C \epsilon \mathcal{M}_1 \cap \mathcal{M}_2$

$E_i \epsilon \mathcal{M}_1 \cap \mathcal{M}_2 \Rightarrow E_i \epsilon \mathcal{M}_1$ and $E_i \epsilon \mathcal{M}_2 \Rightarrow \bigcup E_i \epsilon \mathcal{M}_1 \cap \mathcal{M}_2$

iii) Let $\mathcal{M}_1 = \mathcal{P}(S)$ then $\mathcal{M}_1$ is a σ-algebra.

Let $\mathcal{M}_2 = \{\phi, S\}$ then $\mathcal{M}_2$ is a σ-algebra.

But $\mathcal{M}_1 - \mathcal{M}_2$ is not a σ-algebra since it does not contain ϕ.