Question

Show that $|\sin z|^2 = \sin^2 x + \sinh^2 y$ where $z = x + iy$ and hence that, for any positive integer n,

i) on the lines $y = \pm(n + \frac{1}{2})$, $\quad |\csc \pi z|^2 \leq \operatorname{csch}^2(\frac{\pi}{2})$

ii) on the lines $x = \pm(n + \frac{1}{2})$, $\quad |\csc \pi z|^2 \leq 1$.

The above lines form the sides of a square Γ_n. Prove that

$$\lim_{n\to\infty} \int_{\Gamma_n} \frac{\pi \csc \pi z}{z^2} dz = 0$$

and, using the calculus of residues, deduce that

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r^2} = \frac{\pi^2}{12}.$$

Answer

$\sin(x + iy) = \sin x \cos iy + \cos x + \sin iy = \sin x \cosh y + i \cos x \sinh y$

$|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$

$\qquad = \sin^2 x \cosh^2 y + (1 - \sin^2 x) \sinh^y$

$\qquad = \sin^2 x(\cosh^2 y - \sinh^2 y) + \sinh^2 y = \sin^2 x + \sinh^2 y$

Now if $y = \pm(n + \frac{1}{2})$, $\quad |\sin \pi z|^2 \geq \sinh^2 \pi y$

$= \sinh^2(\pm(n + \frac{1}{2})\pi \geq \sinh^2 \frac{1}{2}\pi$. So $|\csc \pi z|^2 \leq \operatorname{csch}^2(\frac{1}{2}\pi)$

If $x = \pm(n + \frac{1}{2})$, $\quad |\sin^2 \pi z| \geq \sin^2 \pi x = \sin^2(\pm(n + \frac{1}{2})\pi) = 1$

So $|\csc^2 \pi z| \leq 1$

Now on the square Γ_n, $|z| \geq n + \frac{1}{2}$ and $|csc^2 \pi z| \leq \max(1, \operatorname{csch}^2 \frac{1}{2}\pi) = K^2$

So $\left| \int_{\Gamma_n} \frac{\pi \csc \pi z}{z^2} dz \right| \leq \frac{\pi k 8(n + \frac{1}{2})}{(n + \frac{1}{2})^2} \to 0$ as $n \to \infty$.

Now at $z = n \neq 0$, $\frac{\pi \csc \pi z}{z^2}$ has residue $\frac{(-1)^n}{n^2}$.

At $z = 0$, $\frac{\pi \csc \pi z}{z^2}$ has a pole of order 3, with residue $\frac{\pi^2}{3!}$

So $\int_{\Gamma_n} \frac{\pi \csc \pi z}{z^2} = 2\pi i \left(\frac{\pi^2}{3!} + \sum_{-n,\, n \neq 0}^{n} \frac{(-1)^r}{r^2} \right)$

thus $\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r^2} = \frac{\pi^2}{12}$