**Question**

Use the calculus of residues to show that

a) $\displaystyle\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$

b) $\displaystyle\int_0^\infty \frac{x^{\frac{1}{2}}}{(1+x)^2}dx = \frac{\pi}{2}.$

**Answer**

a) Let $f(z) = \dfrac{1}{1+z^4}$, this has simple poles at the fourth roots of $-1$,

i.e. $\dfrac{\pm 1 \pm i}{\sqrt{2}}$

Using $\Gamma$, the poles inside $\Gamma$ are at $\dfrac{1+i}{\sqrt{2}}$ and $\dfrac{-1+i}{\sqrt{2}}$ $(R > 1)$

$\text{Res}\left(\dfrac{1+i}{\sqrt{2}}\right) = \dfrac{-1-i}{4\sqrt{2}}$ $\qquad$ $\text{Res}\left(\dfrac{-1+i}{\sqrt{2}}\right) = \dfrac{1-i}{4\sqrt{2}}$

$$\int_\Gamma f(z)dz = 2\pi i\left(\frac{-1-i}{4\sqrt{2}} + \frac{1-i}{4\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}$$

$$\left|\int_{\text{semicircle}} \frac{1}{1+z^4}dz\right| \leq \frac{\pi R}{R^4 - 1} \to 0 \text{ as } R \to \infty.$$

Thus $\displaystyle\int_{-\infty}^\infty \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}$ and so $\displaystyle\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$

b) Let $y = x^{\frac{1}{2}}$ $\qquad$ $y > 0$ so $x = y^2$ and $dx = 2ydy$

$$\int_0^\infty \frac{x^{\frac{1}{2}}}{(1+x)^2}dx = \int_0^\infty \frac{2y^2}{(1+y^2)^2}dy$$

Let $f(z) = \dfrac{2z^2}{(1+z^2)^2}dz$, this has a pole of order 2 at $z = i$,

with residue $\dfrac{-i}{2}$, using diffn formula.

So using $\Gamma$ with $R > 1$,

$\int_\Gamma f(z)dz = 2\pi i \left(\frac{-i}{2}\right) = \pi$

Now on the semicircle $|f(z)| \leq \frac{2R^2}{(R^2-1)^2}$

So $\left|\int_{\text{semicircle}} f(Z)dz\right| \leq \frac{\pi R 2R^2}{(R^2-1)^2} \to 0$ as $R \to \infty$.

Thus we have $\int_{-\infty}^{\infty} \frac{2y^2}{(1+y^2)^2}dy = \pi$

and so $\int_0^\infty \frac{x^{\frac{1}{2}}}{(1+x)^2}dx = \frac{\pi}{2}$.