Question

Show that, for z in the upper half plane, $|\exp iz| \leq 1$.
Show that the function

$$f(z) = \frac{1 + iz - \exp iz}{z^2}$$

has a removable singularity at $z = 0$.
Apply Cauchy's theorem to $f(z)$ using the contour formed by the real axis from $-R$ to R and the upper half of the circle $|z| = R$ and, by letting $R \to \infty$, prove that

$$\int_0^\infty \frac{1 - \cos x}{x^2} dx = \frac{\pi}{2}.$$

Answer

$\exp(iz) = \exp ix \exp -y$
So $|\exp(iz)| = \exp -y \leq 1$ for $y \geq 0$
$f(z) = \dfrac{1 + iz - \exp iz}{z^2} = \dfrac{-z^2 - iz^3 + z^4 \cdots}{z^2}$
$\quad = -1 - iz + z^2 \cdots \to -1$ as $z \to 0$
So $f(z)$ when defined at $z = 0$, by $f(0) = -1$ is analytic.
Thus $\displaystyle\int_C f(z)dz = 0$
$\displaystyle\int_{-R}^R f(x)dx = -\int_{\text{semicircle}} f(z)dz = -\int \frac{1}{z^2} - \int \frac{iz}{z^2} - \int \frac{\exp iz}{z^2} = I_1 + I_2 + I_3$
$|I_1| \leq \dfrac{1}{R^2}\pi R \to 0$ as $R \to \infty$
$|I_3| \leq \dfrac{1}{R^2}\pi R \to 0$ as $R \to \infty$ as $|\exp iz| \leq 1$ on C.
$I_2 = -i\displaystyle\int \frac{1}{z}dz = -i\int_0^\pi \frac{iRe^{i\theta}}{Re^{i\theta}} = \pi$
$\displaystyle\int_{-R}^R f(x)dx = \int_{-R}^R \frac{1 + ix - \cos x - i\sin x}{x^2}dx = 2\int_0^R \frac{1 - \cos x}{x^2}dx$
So $\displaystyle\int_0^\infty \frac{1 - \cos x}{x^2}dx = \frac{\pi}{2}$