Question

Find the real and imaginary parts of the function $\sin z$ where $z = x + iy$. Describe the images of the lines y=constant under the transformation $w = \sin z$. Show that the transformation maps the infinite strip

$$-\frac{\pi}{2} < x < \frac{\pi}{2}, \quad y > 0$$

conformally onto the upper half plane.

Find a conformal transformation which maps the strip conformally onto the inside of the unit circle.

Answer

$\sin z = \sin(x + iy) = \sin x \cos iy + \cos x \sin iy$

$\quad = \sin x \cosh y + i \cos x \sinh iy$

$y = \text{constant} \ \rightarrow w = k_1 \sin x + ik_2 \cos x$ - ellipse

$x = -\frac{\pi}{2} \Rightarrow w = -\cosh y \quad y > 0$ so $-\infty < w < -1$ real

$y = 0 \Rightarrow w = \sin x \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ so $-1 \leq w \leq 1$ real

$x = \frac{\pi}{2} \Rightarrow w = \cosh y \quad y > 0$ so $1 < w < \infty$ real

Thus $w = \sin z$ maps the boundary of S to the real axis. $z = i \Rightarrow w = i \sinh 1$ which is in U. So $w = \sin z$ maps S conformally onto U.

Now $w = \dfrac{z - i}{z + i}$ maps U to D.

So $w = \dfrac{\sin z - i}{\sin z + i}$ maps S to D.