QUESTION

A function $f(x)$ is defined by

$$f(x) = \frac{x^2-2}{x^2-1}$$

(i) Determine the nature of any stationary points of the function.

(ii) Sketch the graph of $f(x)$ showing clearly any stationary points and asymptotes.

[Note: All the working necessary to obtain your answer must be clearly shown.]

ANSWER

$f(x) = \dfrac{x^2-2}{x^2-1}$

(i) $\dfrac{df}{dx} = \dfrac{(x^2-1)(2x)-(x^2-2)(2x)}{(x^2-1)^2} = \dfrac{2x(x^2-1-x^2+2)}{(x^2-1)^2} = \dfrac{2x}{(x^2-1)^2}$

Hence there is only one stationary point, at $x=0$

$$\begin{aligned}\frac{d^2f}{dx^2} &= \frac{(x^2-1)^2 2 - 2x(2(x^2-1)(2x))}{(x^2-1)^4}\\ &= \frac{(x^2-1(2(x^2-1)-8x^2)}{(x^2-1)^4}\\ &= \frac{(x^2-1)(-6x^2-2)}{(x^2-1)^4}\\ &= -2\frac{(1+3x^2)}{(x^2-1)^3}\end{aligned}$$

$\left.\dfrac{d^2f}{dx^2}\right|_{x=0} = -\dfrac{2}{(-1)^3} = 2 > 0$ therefore $x=0$ is a minimum.

(ii) Clearly f becomes undefined when $x^2-1=0$, so $x=\pm1$ are the asymptotes. The function is even, since $f(-x) = \dfrac{(-x)^2-2}{(-x)^2-1} = \dfrac{x^2-2}{x^2-1} = f(x)$

$f(0) = \dfrac{-2}{-1} = 2$ therefore the minimum value of f is 2.

$f(x) = 0$ when $x^2 = 2,\ x = \pm\sqrt{2}$

As $x \to +\infty,\ f(x) = \dfrac{1 - \frac{2}{x^2}}{1 - \frac{1}{x^2}} \to \dfrac{1}{1} = 1$ asymptote.
As $x \to -\infty, f \to 1$ (same as above, or by the symmetry of even functions)
$\dfrac{d^2 f}{dx^2} = -\dfrac{2(1+3x^2)}{(x^2-1)^3} \neq 0$ for any x, therefore there is no point of inflexion.

$x = 1^+,\ f = \dfrac{(-)}{(+)} = (-)$

$x = 1^-,\ f = \dfrac{(-)}{(-)} = (+)$

$f = \dfrac{x^2-2}{x^2-1} = \dfrac{x^2-1-1}{x^2-1} = 1 - \dfrac{1}{x^2-1}$ therefore $f < 1$ as $x \to +\infty$.

We can complete the graph since f is even (symmetric about the y-axis).