QUESTION

(a) The probability density function of the random variable X is given by

$$f_X(x) = \begin{cases} e^{-2cx} \text{ for } x \geq 0, & \\ 0 & \text{otherwise.} \end{cases}$$

(i) Find the value of c.

(ii) Find the mean of X.

(iii) Show that the distribution function is $F_X(x) = 1 - e^{-x}$.

(b) A company produces a video tape which lasts, on average, for 150 hours with a standard deviation of 36 hours. Assuming that the life of a tape is normally distributed, find the probability that any given tape will last less than 96 hours.

ANSWER

(a) **(i)** $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$ therefore

$$\begin{aligned} \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} e^{-2cx}\,dx &= \left[\frac{e^{-2cx}}{-2c}\right]_0^\infty \\ &= 0 - \left(\frac{1}{-2c}\right) = \frac{1}{2c} = 1, \\ c &= \frac{1}{2} \end{aligned}$$

(ii)

$$\begin{aligned} \text{Mean of } X &= \mu_X = \int_{-\infty}^{\infty} x f_X(x)\,dx \\ &= \int_0^x x e^{-x}\,dx \\ &= \left[x\frac{e^{-x}}{-1}\right]_0^\infty - \int_0^\infty \left(-e^{-x}\right).1\,dx \\ &= 0 - 0 + \left[\frac{e^{-x}}{-1}\right]_0^\infty \\ &= 0 - (0 - 1) = 1 \end{aligned}$$

(iii)

$$\begin{aligned}F_X(x) &= \int_{-\infty}^{x} f_X(x)\,dx = \int_0^x e^{-x}\,dx \\ &= \left[\frac{e^{-x}}{-1}\right]_0^x = -e^{-x} - (-1) \\ &= 1 - e^{-x}\end{aligned}$$

(b) We are given that $\mu_X = 150$, $\sigma_X = 36$. Put $Z = \dfrac{X - \mu}{36}$ then $Z \sim N(0, 1)$

$$\begin{aligned}P(X < 96) &= P\left(Z < \frac{96 - 150}{36}\right) \\ &= P(Z < -1.5) \\ &= P(Z > 1.5), \text{ by symmetry} \\ &= 1 - P(Z \leq 1.5) \\ &= 1 - 0.9332 = 0.0668\end{aligned}$$