QUESTION

(i) Use the elimination method to find the inverse of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & -1 \\ 3 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

and verify that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

(ii) The inverse of any non-singular matrix $\mathbf{C}$ satisfies the equations $\mathbf{C}\mathbf{C}^{-1} = \mathbf{C}^{-1}\mathbf{C} = \mathbf{I}$. Deduce that

$$(\mathbf{C}^{-1})^T = (\mathbf{C}^T)^{-1}.$$

(iii) Using parts (i) and (ii), or otherwise, solve the equations $\mathbf{A}^T\mathbf{X}{=}\mathbf{b}$, where $\mathbf{A}$ is the matrix defined in (i) and $\mathbf{b} = \begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix}$.

ANSWER

(i) $$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right),$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 4 & -3 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right), \text{ row 2} - 3\times\text{row 1}$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 4 & 3 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right), (-1)\times\text{row 2}$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & -2 & 1 & 0 \\ 0 & 1 & -4 & 3 & -1 & 0 \\ 0 & 0 & 4 & -2 & 1 & 1 \end{array}\right), \text{ row 1} - \text{ row 2, row 3} - \text{ row 2}$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & -2 & 1 & 0 \\ 0 & 1 & -4 & 3 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{array}\right), \ (\tfrac{1}{4})\times\text{row 3}$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{array}\right), \text{ row 1} - 3\times\text{row 3, row 2} + 4\times\text{row 3}$$

Therefore $A^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 1 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$

$AA^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 3 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ 1 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(ii) $CC^{-1} = I$ therefore $(CC^{-1})^T = I^T;\ (C^{-1})^T C^T = I$
$C^{-1}C = I$ therefore $(C^{-1}C)^T = I^T;\ (C^T)(C^{-1})^T = I$
Hence $(C^{-1})^T = (C^T)^{-1}$

(iii) $A^T X = \mathbf{b}$, solution is $X = (A^T)^{-1}\mathbf{b} = (A^{-1})^T\mathbf{b}$, using (ii).

Therefore

$$X = \begin{pmatrix} -\frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1}{4} & 0 & \frac{1}{4} \\ -\frac{3}{4} & 1 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -2+1-2 \\ 1+0+1 \\ -3+1+1 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}$$