Question

Find the values of λ and μ for which the following systems of equations is consistent, and find the general solution in all cases.

$$\begin{aligned} x_1+2x_2+x_3+x_4 &= 13 \\ x_1+\lambda x_2+4x_3+6x_4 &= 18 \\ x_1-5x_3-9x_4 &= \lambda \\ x_1+5x_2+10x_3+16x_4 &= \mu \end{aligned}$$

Answer

$$\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 13 \\ 1 & \lambda & 4 & 6 & 18 \\ 1 & 0 & -5 & -9 & \lambda \\ 1 & 5 & 10 & 16 & \mu \end{array} \rightarrow \begin{array}{cccc|c} 1 & 2 & 1 & 1 & 13 \\ 0 & \lambda-2 & 3 & 5 & 5 \\ 0 & -2 & -6 & -10 & \lambda-13 \\ 0 & 3 & 9 & 15 & \mu-13 \end{array}$$

$$\rightarrow \begin{array}{cccc|c} 1 & 2 & 1 & 1 & 13 \\ 0 & \lambda-2 & 3 & 5 & 5 \\ 0 & 3 & 9 & 15 & -\frac{3}{2}(\lambda-13) \\ 0 & 3 & 9 & 15 & \mu-13 \end{array}$$

Must have $\mu-13=-\frac{3}{2}(\lambda-13)$
i.e. $3\lambda+2\mu-65=0$
Then we get rid of row 4

$$\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 13 \\ 0 & \lambda-2 & 3 & 5 & 5 \\ 0 & 3 & 9 & 15 & -\frac{3}{2}(\lambda-13) \end{array}$$

$\lambda=3$ gives both sides of row 3 to be zero.
$\lambda\neq 3$ gives $x_2=\frac{1}{2}$

$\lambda=3$ gives $\mu=28$

Thus $\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 13 \\ & 1 & 3 & 5 & 5 \end{array}$

Solution is

$$\begin{pmatrix} 3 \\ 5 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 5 \\ -3 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 9 \\ -5 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda \neq 3$ gives $x_2 = \frac{1}{2}$ and solution

$$\begin{pmatrix} 10 + \frac{\lambda}{6} \\ \frac{1}{2} \\ 2 - \frac{\lambda}{6} \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} \frac{2}{3} \\ 0 \\ -\frac{5}{3} \\ 1 \end{pmatrix}$$