Question

(i) Verify the identity

$$z^5 - 1 = (z-1)z^2\left\{\left(z+\frac{1}{z}\right)^2 + \left(z+\frac{1}{z}\right) - 1\right\}$$

and putting $z = \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)$ and evaluate $\cos\left(\frac{2\pi}{5}\right)$

(ii) If $w = \cosh z$ show that the lines $x = x_0$, $y = y_0$ correspond the ellipse and hyperbolas

$$\frac{u^2}{\cosh^2 x_0} + \frac{v^2}{\sinh^2 x_0} = 1, \quad \frac{u^2}{\cos^2 y_0} - \frac{v^2}{\sin^2 y_0} = 1$$

Also show that the whole of the w-plane corresponds to any strip of the z-plane of the π bounded by lines parallel to the x-axis.

Answer

(i) $(z-1)z^2 \times \left\{\left(z+\frac{1}{z}\right)^2 \left(z+\frac{1}{z}\right) - 1\right\}$

$= (z-1)z^2\left\{z^2 + 2 + \frac{1}{z^2} + z + \frac{1}{z} - 1\right\}$

$= (z-1)(z^4 + z^3 + z^2 + z + 1) = z^5 - 1$

Putting $z = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$ $\quad z^5 - 1 = 0$ and $z + \frac{1}{z} = 2\cos\frac{2\pi}{5}$

So $4c^2+2c-1 = 0$ and $c > 0$ which gives $c = \dfrac{\sqrt{5}-1}{4} = 0.309016994372$

(ii) $w = \cosh z$ so $u + iv = \cosh(x+iy) = \cosh x \cos y + i \sinh x \sin y$

If $x = x_0$ $\quad \dfrac{u^2}{\cosh^2 x_0} + \dfrac{v^2}{\sinh^2 x_0} = \cos^2 y + \sin^2 y = 1$

If $y = y_0$ $\quad \dfrac{u^2}{\cos^2 y_0} + \dfrac{v^2}{\sin^2 y_0} = \cosh^2 x + \sinh^2 x = 1$

The system of hyperbolas covers the w-plane and $\cos^2 y_0$ and $\sin^2 y_0$ are periodic with period π. So any strip parallel to the x-axis of width π will map onto the whole of the w-plane.