Question
Verify the following integral.
$\int_0^\infty \frac{dx x^{-p}}{x-1} = \pi \cot p\pi, \ 0 < p < 1$
Answer
The integral converges for $0 < \rho < 1$ only if we can treat the singularity (pole) of the integrand at $x = +1$ in a suitable way.
This is done by considering:

$$J = \int_C \frac{dz z^{-\rho}}{(z-\alpha)}$$

where we take C as PICTURE

for $z \in \mathbf{C}, \ \alpha \in \mathbf{C}$
Now $J = 2\pi i \times (\text{residue at } \alpha) = 2\pi \mathrm{i}\alpha^{-\rho}$
$-2\pi i\alpha^{-\rho} = \int_\epsilon^R + \int_{\Gamma_2} + \int_{Re^{2\pi i}}^\epsilon + \int_{\Gamma_1}$
Take limits $R \to \infty, \ \epsilon \to 0$ and $\int_{\Gamma_1}, \ \int_{\Gamma_2} \to 0$
Therefore

$$\begin{aligned} 2\pi i\alpha^{-\rho} &= \underbrace{\int_0^\infty \frac{dx \ x^{-\rho}}{(x-\alpha)}}_{z=x} + \underbrace{\int_{\infty e^{2\pi i}}^0 \frac{dz \ z^{-\rho}}{(z-\alpha)}}_{z=xe^{2\pi i}} \\ &= \int_0^\infty \frac{dx \ x^{-\rho}}{(x-\alpha)} + \int_\infty^0 \frac{d(xe^{2\pi i})(xe^{2\pi i})^{-\rho}}{(xe^{2\pi i}-\alpha)} \\ 2\pi i\alpha^{-\rho} &= \int_0^\infty \frac{dx \ x^{-\rho}}{(x-\alpha)}(1-e^{-2\pi i}) \end{aligned}$$

So $\int_0^\infty \frac{dx \ x^{-\rho}}{(x-\alpha)} = \frac{2\pi i\alpha^{-\rho}}{1-e^{-2\pi i}}, \quad \alpha \notin \mathbf{R}+$

When $\alpha > 0$ it can be either above the cut or below. If above we have

$$\alpha = |\alpha| e^{i0}$$

If below we have

$$\alpha = |\alpha| e^{2\pi i}$$

Let $|\alpha| = 1$ now. Then:

$\underline{\alpha = e^{i0}}$: $\displaystyle\int_0^\infty \frac{dx\ x^{-\rho}}{(x - e^{i0})} = \frac{2\pi i (e^{i0})^{-\rho}}{1 - e^{-2\pi i\rho}}$

$\underline{\alpha = e^{2\pi i}}$: $\displaystyle\int_0^\infty \frac{dx\ x^{-\rho}}{(x - e^{2\pi i})} = \frac{2\pi i (e^{2\pi i})^{-\rho}}{1 - e^{-2\pi i\rho}}$

What's the answer? It's natural to take the average. This is called Cauchy's Principal Value and is sometimes written as

$$\begin{aligned}
\int_0^\infty \frac{dx\ x^{-\rho}}{x-1} &= \frac{2\pi i}{1 - e^{-2\pi i\rho}}(1 + e^{-2\pi i\rho}) \times \frac{1}{2} \text{ (average)} \\
& \quad \text{where x} \neq 1 \\
&= \frac{2\pi}{2} \times \frac{e^{-i\pi\rho}}{e^{-i\pi\rho}} \frac{(e^{i\pi\rho} + e^{-i\pi\rho})}{(e^{i\pi\rho} - e^{-i\pi\rho})} \frac{2i}{2} \\
&= \pi \frac{\cos \pi\rho}{\sin \pi\rho}
\end{aligned}$$

$$\Rightarrow \int_0^\infty \frac{dx\ x^{-\rho}}{x-1} = \pi \cot \pi\rho, \quad x \neq 1$$