Question

Verify the following integral.

$$\int_0^{\infty} \frac{dx}{(x+1)\sqrt{x}} = \pi$$

Answer

$J = \displaystyle\int_C \frac{dz}{(z+1)\sqrt{z}}$ where C is the contour

PICTURE

Pole at $z = -1$. Branch cut from branch point at $z = 0$ chosen to lie along $0 \to \infty$.

$J = 2\pi i \times \text{residue at z} = -1) = 2\pi\text{i} \cdot \dfrac{1}{\sqrt{-1}} = 2\pi$

Now

$J = \displaystyle\int_{\substack{\Gamma_1 \\ radius\ R}} + \int_{\substack{\Gamma_2 \\ radius\ \epsilon}} + \int_{\Gamma_3} + \int_{\Gamma_4} = 2\pi$

Now $\displaystyle\int_{\Gamma_1}$ and $\displaystyle\int_{\Gamma_2} \to 0$ as $R \to \infty,\ \epsilon \to 0$

$$\begin{aligned}
J &= \underbrace{\int_0^{\infty} \frac{dz}{(z+1)\sqrt{z}}}_{z = x} + \underbrace{\int_{+\infty}^{0} \frac{dz}{(z+1)\sqrt{z}}}_{z = xe^{-2\pi i}} \\
&= \int_0^{\infty} \frac{dx}{(x+1)\sqrt{x}} + \int_{\infty}^{0} \frac{dxe^{-2\pi i}}{(x+1)e^{i\pi}\sqrt{x}} \\
&= 2\int_0^{\infty} \frac{dx}{(x+1)\sqrt{x}} \\
&= 2\pi \\
&\Rightarrow \underline{\int_0^{\infty} \frac{dx}{(x+1)\sqrt{x}} = \pi}
\end{aligned}$$