NOTE REFERENCE TO QUESTION 4

Question

Verify the following integral.

$\int_0 \infty \, dx \frac{(\log x)^2}{x^4+1} = \frac{3\pi^3\sqrt{2}}{16}$

Answer

Use contour of Q4.

$J = \oint_C \frac{dz \ (\log z)^2}{(z^4+1)}$

$$
\begin{aligned}
J &= 2\pi i \times [\mathrm{res}(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}) + \mathrm{res}((\mathrm{e}^{\frac{3\mathrm{i}\pi}{4}}))] \\
&= 2\pi i \times \left\{ \lim_{z \to e^{\frac{i\pi}{4}}} \left[\frac{(z - e^{\frac{i\pi}{4}})(\log z)^2}{(z^4+1)} \right] \right. \\
&\quad \left. + \lim_{z \to e^{\frac{3i\pi}{4}}} \left[\frac{(z - e^{\frac{3i\pi}{4}})(\log z)^2}{(z^4+1)^2} \right] \right\} \\
&\quad \text{use l'Hopital} \\
&= 2\pi i \left\{ \frac{[\mathrm{Log}(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}})]^2}{4e^{\frac{3i\pi}{4}}} + \frac{[\mathrm{Log}(\mathrm{e}^{\frac{3\mathrm{i}\pi}{4}})]^2}{4e^{\frac{9i\pi}{4}}} \right\} \\
&= \frac{2\pi i}{4} e^{\frac{-3i\pi}{4}} \left\{ \left(\frac{i\pi}{4}\right)^2 + \left(\frac{3i\pi}{4}\right)^2 i \right\} \\
&= \frac{\pi i}{2} \frac{(-1-i)}{\sqrt{2}} (i + 9i) \left(-\frac{\pi^2}{16} \right) \\
&= \frac{-5\pi^3\sqrt{2}}{32} - \frac{i\pi^3}{8}\sqrt{2}
\end{aligned}
$$

Now $J = \int_{-R}^{-\epsilon} + \int_{\Gamma_1} + \int_{\epsilon}^{R} + \int_{\Gamma_2}$

Take limit as $R \to \infty, \ \epsilon \to 0 \ \ ds \int_{\Gamma_1}$ and $\int_{\Gamma_2} \to 0$

Thus

$$\int_{\infty}^{0} \frac{d(xe^{i\pi})}{[(xe^{i\pi})^4+1]} \underbrace{[\log(xe^{i\pi})]^2}_{[\log x + i\pi]^2} + \int_0^{\infty} \frac{dx \ (\log x)^2}{x^4+1} = \frac{-\pi^3\sqrt{2}}{32} - \frac{i\pi^3\sqrt{2}}{8}$$

$$\Rightarrow 2\int_0^{\infty} \frac{dx \ \log^2 x}{x^4+1} + 2\pi i \int_0^{\infty} \frac{dx \ \log x}{(x^4+1)} - \pi^2 \int_0^{\infty} \frac{dx}{x^4+1} = \frac{-5}{32}\pi^3\sqrt{2} - \frac{i\pi^3}{8}\sqrt{2}$$

So

$$2\int_0^\infty \frac{dx\ \log^2 x}{x^4+1} + 2\pi i \times (Q4) - \pi^2 \frac{\pi}{2\sqrt{2}}\ \text{(see Q4)} = \frac{-5\pi^3\sqrt{2}}{32} - \frac{\mathrm{i}\pi^3\sqrt{2}}{8}$$

$$2\int_0^\infty \frac{dx\ (\log x)^2}{x^4+1} + 2\pi i \times \left(\frac{-\pi^2\sqrt{2}}{16}\right) - \frac{\pi^3}{2\sqrt{2}} = \frac{-5\pi^3\sqrt{2}}{32} - \frac{i\pi^3\sqrt{2}}{8}$$

$$2\int_o^\infty \frac{dx\ (\log x)^2}{x^4+1} - i\frac{\pi^3\sqrt{2}}{8} - \frac{\pi^3}{2\sqrt{2}} = \frac{-5\pi^3\sqrt{2}}{32} - \frac{i\pi^3\sqrt{2}}{8}$$

$$\Rightarrow 2\int_0^\infty \frac{dx\ (\log x)^2}{x^4+1} = \frac{3}{32}\pi^3\sqrt{2}$$

$$\Rightarrow \int_0^\infty \frac{dx\ (\log x)^2}{(x^4+1)} = \frac{3\pi^3\sqrt{2}}{64}$$