Question

Verify the following integral.

$$\int_0^\infty dx \frac{\log x}{x^4+1} = -\frac{\pi^2\sqrt{2}}{16}$$

Answer

Consider $J = \oint_C \frac{dz \log z}{(z^4+1)}$ where C is

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simple poles at $z^4+1=0 \Rightarrow z = e^{\frac{i\pi}{4}},\ e^{\frac{3i\pi}{4}},\ e^{\frac{5i\pi}{4}},\ e^{\frac{7i\pi}{4}}$

Branch points at $z=0$.

Set cut along negative imaginary axis only $e^{\frac{i\pi}{4}}$ and $e^{\frac{3i\pi}{4}}$ count.

$$\begin{aligned}
J &= 2\pi i \times [\mathrm{res}(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}) + \mathrm{res}(\mathrm{e}^{\frac{3\mathrm{i}\pi}{4}})] \\
&= 2\pi i \times \left\{ \lim_{z\to e^{\frac{i\pi}{4}}} \left[\frac{(z-e^{\frac{i\pi}{4}})}{(z^4+1)} \log z \right] + \lim_{z\to e^{\frac{3i\pi}{4}}} \left[\frac{(z-e^{\frac{3i\pi}{4}})\log z}{(z^4+1)} \right] \right\} \\
&\quad \text{use l'Hopital} \\
&= 2\pi i \left\{ \frac{1}{4e^{\frac{3i\pi}{4}}} \log(e^{\frac{3i\pi}{4}}) + \frac{1}{4e^{\frac{9i\pi}{4}}} \log(e^{\frac{3i\pi}{4}}) \right\} \\
&= \frac{2\pi i}{4} e^{\frac{-3i\pi}{4}} \left\{ \frac{i\pi}{4} + \frac{3i\pi}{4} e^{\frac{-6i\pi}{4}} \right\} \\
&\quad \text{since } \mathrm{e}^{\frac{-3\mathrm{i}\pi}{2}} = +\mathrm{i} \\
&= \frac{-\pi^2}{8} \frac{(-1-i)}{\sqrt{2}} (1+3i) \\
&= \frac{-\pi^2}{8} \frac{(2-4i)}{\sqrt{2}} \\
&= \frac{-\pi^2\sqrt{2}}{8} + \frac{\pi^2 i}{2\sqrt{2}}
\end{aligned}$$

Now $J = \int_{-R}^{-\epsilon} + \int_{\Gamma_1} + \int_{\epsilon}^{R} + \int_{\Gamma_2} = \dfrac{-\pi^2\sqrt{2}}{8} + \dfrac{\pi^2 i}{2\sqrt{2}}$

Take limit at $R \to \infty, \epsilon \to 0$ $\quad \int_{\Gamma_1}$ and $\int_{\Gamma_2} \to 0$

Thus

$$\underbrace{\int_{\infty}^{0} \frac{d(xe^{i\pi})\ \log(xe^{i\pi})}{(xe^{i\pi})^4+1}}_{\text{setting } z = xe^{i\pi}} + \int_0^\infty \underbrace{\frac{dx\ \log x}{x^4+1}}_{z = x} = \frac{-\pi^2\sqrt{2}}{8} + \frac{\pi^2 i}{2\sqrt{2}}$$

(consistent with cut along $\arg(z) = -\frac{\pi}{2}$)

$$-\int_{\infty}^{0} \frac{dx\ \log x}{x^4+1} + i\pi \int_0^\infty \frac{dx}{x^4+1} + \int_0^\infty \frac{dx\ \log x}{x^4+1} = \frac{-\pi^2\sqrt{2}}{8} + \frac{\pi^2 i}{2\sqrt{2}}$$

$$\Rightarrow 2\int_0^\infty \frac{dx\ \log x}{x^4+1} + i\pi \int_0^\infty \underbrace{\frac{dx}{x^4+1}}_{= \frac{\pi}{2\sqrt{2}}} = \frac{-\pi^2\sqrt{2}}{8} + \frac{\pi^2 i}{2\sqrt{2}}$$

(poles, simple at $x = e^{i\frac{\pi}{4}},\ x = e^{i\frac{3\pi}{4}}$ and using a D-shaped contour)

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$$\Rightarrow \int_0^\infty \frac{dx\ \log x}{x^4+1} = \frac{-\pi^2\sqrt{2}}{16} + \frac{\pi^2 i}{4\sqrt{2}} - \frac{\pi^2 i}{4\sqrt{2}} = \frac{-\pi^2\sqrt{2}}{16}$$