Exam Question
Topic: LaplaceODE
Suppose that

$$\begin{aligned} 5y_1'' - 10y_1' - 13y_2' + 11y_1 + 13y_2 &= 0 \\ y_2'' + 13y_1' - 2y_2' - 13y_1 - 29y_2 &= 0, \end{aligned}$$

and that $y_1(0) = 1, \quad y_1'(0) = 4, \quad y_2(0) = 1, \quad y_2'(0) = 3.$
Show that $\bar{y}_1 = \dfrac{p^3 + 6p - 19}{(p^2 - 2p - 3)(p^2 - 2p + 10)}$ and hence find $y_1(x)$.

Solution
Transforming the differential equations gives

$$\begin{aligned} &5(p^2\bar{y}_1 - p - 4) - 10(p\bar{y}_1 - 1) - 13(\bar{y}_2 - 1) + 11\bar{y}_1 + 13\bar{y}_2 = 0 \\ &\Rightarrow (5p^2 - 10p + 11)\bar{y}_1 + (-13p + 13)\bar{y}_2 = 5p - 3 \qquad (1) \\ &(p^2\bar{y}_2 - p - 3) + 13(p\bar{y}_1 - 1) - 2(p\bar{y}_2 - 1) - 13\bar{y}_1 - 29\bar{y}_2 = 0 \\ &\Rightarrow (13p - 13)\bar{y}_1 + (p^2 - 2p - 29)\bar{y}_2 = p + 14 \qquad (2) \end{aligned}$$

Now $(p^2 - 2p - 29) \times (1) - (-13p + 13) \times (2)$ gives

$$\begin{aligned} &5(p^4 - 4p^3 + 11p^2 - 14p - 30)\bar{y}_1 = 5(p^3 + 6p - 19) \\ &\Rightarrow \bar{y}_1 = \frac{p^3 + 6p - 19}{(p^2 - 2p - 3)(p^2 - 2p + 10)} \\ &\Rightarrow \bar{y}_1 = \frac{p - 1}{(p - 1)^2 - 2^2} + \frac{3}{(p - 1)^2 + 3^2} \\ &\Rightarrow y_1(x) = \mathrm{e}^{-x}(\cosh 2x + \sin 3x). \end{aligned}$$