Exam Question
Topic: LaplaceODE
Find the solution of the differential equation

$$y'' + y = r(x),$$

where

$$r(x) = \begin{cases} 0 & \text{if } 0 < x < 2; \\ 1 & \text{otherwise,} \end{cases}$$

and where $y(0) = 1; \quad y'(0) = 0.$

Solution
Using the Heaviside function gives $r(x) = H(x-2)$.
Transforming the differential equation gives $p^2\bar{y} - p + \bar{y} = \dfrac{\mathrm{e}^{-2p}}{p}$.

$$\begin{aligned} \bar{y} &= \frac{\mathrm{e}^{-2p}}{p(p^2+1)} + \frac{p}{p^2+1} \\ &= \frac{\mathrm{e}^{-2p}}{p} - \frac{\mathrm{e}^{-2p}.p}{p^2+1} + \frac{p}{p^2+1} \\ y &= H(x-2) - H(x-2)\cos(x-2) + \cos x. \end{aligned}$$