QUESTION

(a) Show that every subgroup of index 2 in any group G must be a normal subgroup, and that every subgroup of an abelian group is also normal. Give an example of a non-normal subgroup of a finite group.

(b) Describe the group of rotations of the cube and regarding them as permutations of the 4 diagonals write them in disjoint cycle notation and say which of them are even.

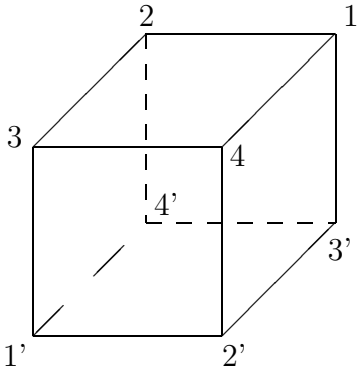
ANSWER

(a) If H has index 2 in G then G is the disjoint union of H and gH for some element $g \in \frac{G}{H}$. Since $g \notin H$ Hg ia also disjoint from H and therefore $Hg = gH$.

If G is abelian then $gh = hg \ \forall g, h \in G$ and in particular $gH = Hg$

Let $G = D_n$ $n \geq 3$ and $g = \sigma_i$ be a reflection. Then $\langle g \rangle$ is not normal since $\rho \langle \sigma_i \rangle \neq \langle \sigma_i \rangle \rho$.

(b) There are 24 rotations of the cube. If the diagonals are labeled 1,2,3,4 anti-clockwise around the top face of the cube.



We have rotations of order 3 around the diagonals given by

anti clockwise $2\frac{\pi}{3}$	clockwise $2\frac{\pi}{3}$
(234)	(243)
(134)	(143)
(124)	(142)
(123)	(132)

These are all even.

Given any edge there is an order 2 rotation around the axis joining the midpoint of the edge to that on the opposite edge. There are 6 such rotations given by

(12) (around the edge with vertices 1,2)
(23) (around the edge with vertices 2,3)
(34) (around the edge with vertices 3,4)
(41) (around the edge with vertices 4,1)
(13) (around the edge with vertices 1,3')
(24) (around the edge with vertices 2,4')

These are all odd.

There are 3 anticlockwise rotations of order 2 around midpoints of faces giving

(1243) around the midpoint of the face {1,2,4',3'}
(1423) around the midpoint of the face {1,4,2',3'}
(1234) around the midpoint of the face {1,2,3,4}

these are all odd as are their inverses (1342), (1324) and (1432) whereas the squares of the rotations give (14)(23), (12)(43) and (13)(24) which are all even. The last rotation is the identity which is even.