QUESTION

Decide for each of the following statements whether or not it is true giving a brief explanation of your answer.

(i) The even permutations in S_n form a subgroup.

(ii) The union of any two subgroups of group G is itself a subgroup of G.

(iii) Every finite group is isomorphic to a subgroup of S_n for some positive integer n.

(iv) For every positive integer n there is a non-abelian group with precisely n elements.

(v) If $f : G \Rightarrow G'$ is an injective homomorphism then the order of G divides the order of G'.

(vi) Every subgroup of an abelian group is abelian.

ANSWER

(i) True

σ_i $i = 1, 2$ are even $\Leftrightarrow$ each can be written as products of even numbers of transpositions. The product is also a product of an even number of transpositions. so is even.

(ii) True

Let $H, K < G$. Then for any $h, k \in H \cap K$ $hk \in H$ and $hk \in K$ so $hk \in H \cap K$, $h^{-1} \in H$ and $h^{-1} \in K$ so $h^{-1} \in H \cap K$ and $e \in H$ and $e \in K$ so $e \in H \cap K$.

(iii) True:

Let $n = |G|$ and choose a bijection f between G and $\{1, \ldots, n\}$. f induces an isomorphism from S_G to S_n.

G embeds in S_G via the left regular representation $g \mapsto (\sigma_g : h \mapsto gh)$.

(iv) False

If p is prime then every group of order p is cyclic and therefore abelian.

(v) True

$f(G)$ is a subgroup of G' so $|f(G)|$ divides $|G'|$.

Since f is injective $|f(G)| + |G|$

(vi) True

Let G be abelian, $H < G$. For any $g, h \in G$ $gh = hg$. In particular, for $g, h \in H,\ gh = hg$.