QUESTION

(a) Define the following terms:

- **(i)** homomorphism,
- **(ii)** kernel,
- **(iii)** isomorphism,
- **(iv)** normal subgroup,
- **(v)** quotient group.

(b) Show that the kernel of a homomorphism is a normal subgroup (you may assume that it is a subgroup), and state and prove the First Isomorphism Theorem.. Illustrate the theorem by using an example of a surjective homomorphism from S_4 to $\mathbf{Z}_2$.

ANSWER

(a) **(i)** A homomorphism is a function $f : G \to H$ between groups G and H, such that $f(gk) = f(g)f(k)$ for every $g, k \in G$.

(ii) The kernel of a homomorphism is the set $\ker(f) = \{g \in G | f(g) = e_H\}$

(iii) An isomorphism is a bijective homomorphism.

(iv) A normal subgroup $H \triangleleft G$ is a subgroup such that $g^{0-1}Hg = H \; \forall g \in G$

(v) The quotient $\frac{G}{N}$ is the group of left cosets $\{gN | g \in G\}$ with $gNg'N = gg'N$.

(b) Let $K = \ker F$ and $g \in G$. Then $g^{-1}Kg = \{g^{-1}kg | f(k) = e_H\}$ so for any element $g^{-1}kg \in g^{-1}Kg$ we have $f(g^{-1}kg) = f(g^{-1})f(k)f(g) = f(g^{-1}e_H f(g) = f(g)^{-1}e_H f(g) = e_H$

The First Isomorphism Theorem

Let $f : G \Rightarrow H$ be a surjective homomorphism. Then $\overline{f} : \begin{array}{c} \frac{G}{\ker f} \Rightarrow H \\ g\ker f \mapsto f(g) \end{array}$ is an isomorphism.

Proof

Let $K = \ker f$. $\overline{f}$ is well defined since if $gK = g'K$ then $\overline{f}(gK) = f(g)$ and $\overline{f}(g'K) = f(g;)$, but $g \in g'K$ so $g = g'k$ for some $k \in \ker f \Rightarrow f(g) = f(g'k) = f(g')f(j) = f(g')e_H = f(g')$ as required.

$\overline{f}$ is a homomorphism since $\overline{f}(g'KgK) = \overline{f}(g'gg^{-1}KgK) = \overline{f}(g'gKK) = \overline{f}(g'gK) = g'g = \overline{f}(g'K)\overline{f}(gK)$

$\overline{f}$ is surjective since f was (for any $h \in H \exists g \in G$ with $f(g) = h$ so $\overline{f}(gK) = h$)

$\overline{f}$ is injective since $\overline{f}(gK = e_H \Leftrightarrow f(g) = e_H \Leftrightarrow g \in K \Leftrightarrow gK = K$.

Let sgn:$S_n \to \mathbf{Z}_2$ denote the sign homomorphism with the kernel A_n so by the theorem $\frac{S_n}{A_N}$ is isomorphic to $\mathbf{Z}_2$. It's elements are A_n and $(12)A_n$ and its multiplication table is

	A_n	$(12)A_n$
A_n	A_n	$(12)A_n$
$(12)A_n$	$(12)A_N$	A_n