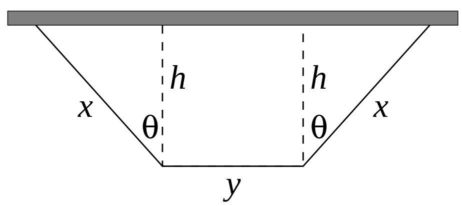
Applications of Partial Differentiation
Extremes within restricted domains

Question
A farmer is constructing an enclosure for some of his animals. He has a 100m long length of straight fence which will be bent at equal angles at equal distances from the ends of the fence. The resulting three segment fence will be placed along an existing fence to make a trapezoid shaped enclosure. What is the maximum area for the enclosure?
Answer
The dimensions of the enclosure are given by



And so $2x + y = 100$ is the length of the fence. In order to obtain maximum area, $x > 0 \longrightarrow (a)$ and $0 < \theta < \pi/2 \longrightarrow (b)$. As $h = x\cos\theta$, this means that the area A is given by

$$\begin{aligned} A &= xy\cos\theta + 2 \times \frac{1}{2}(x\sin\theta)(x\cos\theta) \\ &= x(100-2x)\cos\theta + x^2\sin\theta\cos\theta \\ &= (100x - x2x^2)\cos\theta + \frac{1}{2}x^2\sin 2\theta. \end{aligned}$$

We need a critical point that satisfies conditions (a) and (b), and so

$$\begin{aligned} 0 = \frac{\partial A}{\partial x} &= (100-4x)\cos\theta + x\sin\theta \\ &\Rightarrow \cos\theta(100 - 4x + 2x\sin\theta) = 0 \\ &\Rightarrow 4x - 2x\sin\theta = 100 \\ \Rightarrow x &= \frac{50}{2-\sin\theta} \end{aligned}$$

$$\begin{aligned} 0 = \frac{\partial A}{\partial \theta} &= -(100x - 2x^2)\sin\theta + x^2\cos 2\theta \\ &\Rightarrow x(1 - 2\sin^2\theta) + 2x\sin\theta - 100\sin\theta = 0 \end{aligned}$$

By combining these equations we get

$$\frac{50}{2-\sin\theta}(1-2\sin^2\theta+2\sin\theta)-100\sin\theta=0$$

$$50(1-2\sin^2\theta+2\sin\theta)=100(2\sin\theta-\sin^2\theta)$$

$$50=100\sin\theta$$

And so $\sin\theta = 1/2$, $\Rightarrow \theta = \pi/6$

$$\begin{aligned}\Rightarrow x &= \frac{50}{2-(1/2)}=\frac{100}{3}\\ y &= 100-2x=\frac{100}{3}\end{aligned}$$

This means that the maximum area for the enclosure is

$$A=\left(\frac{100}{3}\right)^2\frac{\sqrt{3}}{2}+\left(\frac{100}{3}\right)^2\frac{1}{2}\frac{\sqrt{3}}{2}=\frac{2500}{\sqrt{3}}$$

square units. And so the three segments of the fences will all be of equal length, with the bend angles at 120^o.