Applications of Partial Differentiation
Extremes within restricted domains

Question
The temperature of a point of a disk D is given by
$T = (x+y)e^{-x^2-y^2}$
Where D is defined as all points such that $x^2 + y^2 \leq 1$.
Find the maximum on minimum values of temperature on D.
Answer
For critical points

$$\begin{aligned} 0 = \frac{\partial T}{\partial x} &= (1-2x(x+y))\, e^{-x^2-y^2} \\ 0 = \frac{\partial T}{\partial y} &= (1-2y(x+y))\, e^{-x^2-y^2} \end{aligned}$$

$$\begin{aligned} \Rightarrow 2x(x+y) &= 1 = 2y(x+y) \\ \Rightarrow x &= y, \quad \text{and } 4x^2 = 1 \\ \Rightarrow x &= y = \pm\frac{1}{2} \end{aligned}$$

The two critical points are then $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2})$. Both of these lie inside the disk, with $T = \pm e^{-1/2}$.
Parameterising on the boundary of the disk gives

$$\begin{aligned} x &= \cos t \\ y &= \sin t \end{aligned}$$

with $0 \leq t \leq 2\pi$.

$$\Rightarrow T = (\cos t + \sin t)e^{-1} = g(t), \quad (0 \leq t \leq 2\pi)$$

Now $g(0) = g(2\pi) = e^{-1}$, so for critical points of g

$$\begin{aligned} 0 = g'(t) &= (\cos t - \sin t)e^{-1} \\ \Rightarrow \tan t &= 1 \\ \Rightarrow t &= \pi/t, \quad \text{or } t = 5\pi/4 \end{aligned}$$

This gives

$$\begin{aligned} g(\pi/t) &= \sqrt{2}e^{-1} \\ \text{and } g(5\pi/4) &= -\sqrt{2}e^{-1} \end{aligned}$$

As $e^{-1/2} > \sqrt{2}e^{-1}$, $\min f = -e^{-1/2}$ and $\max f = e^{-1/2}$.