Applications of Partial Differentiation
Extremes within restricted domains

Question
Find the maximum and minimum values of

$$f(x,y) = xy - x^3y^2$$

Over the square $0 \le x \le 1$, $0 \le y \le 1$.
Answer

$$\begin{aligned} f_1 &= y - 3x^2y^2 = y(1-3x^2y) \\ f_2 &= x - 2x^3y = x(1-2x^2y) \end{aligned}$$

This gives $(0,0)$ as a critical point, with any other critical points having to satisfy $3x^2y = 1$ and $2x^2y = 1$. i.e. $x^2y = 0$.
Therefore $(0,0)$ is the only critical point, and so we need only consider values of f on the boundary as possible limits.
When $x = 0$ or $y = 0$, $f(x,y) = 0$.
When $x = 1$, $f(1,y) = y - y^2 = g(y)$, for $0 \le y \le 1$.
g has a maximum value $1/4$ when $y = 1/2$.
When $y = 1$, $f(x,1) = x - x^3 = h(x)$, for $0 \le x \le 1$.
h has a critical point given by $1 - 3x^2 = 0$; only $x = 1/\sqrt{3}$ is on the side of the square.

$$h\left(\frac{1}{\sqrt{3}}\right) = \frac{2}{3\sqrt{3}} > \frac{1}{4}.$$

On the square, $f(x,y)$ has a minimum value of 0. (At point $(1,10$ and the sides $x = 0$ and $y = 0$.) $f(x,y)$ has a maximum value of $2/(3\sqrt{3})$ (at point $(1/\sqrt{3},1)$, with a smaller local maximum point at $(1,\frac{1}{2})$.