Applications of Partial Differentiation
Extremes within restricted domains

Question
Find the maximum and minimum values of

$$f(x,y) = xy - y^2$$

On the disk $x^2 + y^2 \leq 1$.
Answer
For critical points

$$\begin{aligned} 0 &= f_1(x,y) = y \\ 0 &= f_2(x,y) = x - 2y \end{aligned}$$

So the only critical point is $(0,0)$. This lies inside the disk, with $f(0,0) = 0$.
The boundary of the disk is given by the equations for a circle

$$\begin{aligned} x &= \cos t \\ y &= \sin t \\ \text{with } -\pi \leq t \leq \pi \end{aligned}$$

On this circle

$$\begin{aligned} g(t) &= f(\cos t, \sin t) = \cos t \sin t - \sin^2 t \\ &= \frac{1}{2}[\sin 2t + \cos 2t - 1] \\ g(0) &= g(2\pi) = 0 \\ g'(t) &= \cos 2t - \sin 2t \end{aligned}$$

The critical points of g must satisfy

$$\begin{aligned} \cos 2t &= \sin 2t \\ \text{i.e. } \tan 2t &= 1 \end{aligned}$$

$$\begin{aligned} \text{So } 2t &= \pm\frac{\pi}{4}, \quad \text{or } \pm\frac{5\pi}{4} \\ \Rightarrow t &= \pm\frac{\pi}{8}, \quad \text{or } \pm\frac{5\pi}{8} \end{aligned}$$

This gives

$$g\left(\frac{\pi}{8}\right) = \frac{1}{2\sqrt{2}} - \frac{1}{2} + \frac{1}{2\sqrt{2}}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2}} - \frac{1}{2} > 0 \\
g\left(-\frac{\pi}{8}\right) &= -\frac{1}{2\sqrt{2}} - \frac{1}{2} + \frac{1}{2\sqrt{2}} \\
&= -\frac{1}{2} \\
g\left(\frac{5\pi}{8}\right) &= -\frac{1}{2\sqrt{2}} - \frac{1}{2} - \frac{1}{\sqrt{2}} \\
&= -\frac{1}{2\sqrt{2}} - \frac{1}{2} \\
g\left(-\frac{5\pi}{8}\right) &= \frac{1}{2\sqrt{2}} - \frac{1}{2} - \frac{1}{2\sqrt{2}} \\
&= -\frac{1}{2}
\end{aligned}
$$

So, on the disk

$$
\begin{aligned}
\min(f) &= -\frac{1}{\sqrt{2}} - \frac{1}{2} \\
\max(f) &= \frac{1}{\sqrt{2}} - \frac{1}{2}
\end{aligned}
$$