Applications of Partial Differentiation
Extremes within restricted domains

Question
Find the maximum value of $f(x, y) = \sin x \sin y \sin(x + y)$
Over the triangle bounded by $x = 0$, $y = 0$ and $x + y = \pi$.
Answer
It can easily be seen that $f(x, y) = 0$ on the boundary of the triangle, and that f(x,y)¿0 at all points inside the triangle. So the minimum value of f on the triangle is zero, with the maximum point at some point inside.
For critical points inside the triangle:

$$\begin{aligned} 0 = f_1(x, y) &= \cos x \sin y \sin(x + y) \\ &\quad + \sin x \sin y \cos(x + y) \\ 0 = f_2(x, y) &= \sin x \cos y \sin(x + y) \\ &\quad + \sin x \sin y \cos(x + y) \end{aligned}$$

$$\Rightarrow \cos x \sin y = \cos y \sin x$$

So $x = y$ for critical points inside the triangle, and also that

$$\begin{aligned} \cos x \sin x \sin 2x + \sin^2 x \cos 2x &= 0 \\ 2 \sin^2 x \cos^2 x + 2 \sin^2 \cos^2 x - \sin^2 x &= 0 \\ 4 \cos^2 x &= 1 \end{aligned}$$

$$\begin{aligned} \Rightarrow \cos x &= \pm 1/2 \\ x &= \pm \pi/2 \end{aligned}$$

The critical point inside the triangle is $(\pi/3, \pi/3)$, with $f = 3\sqrt{3}/8$ being the maximum value of f in the triangle.