Applications of Partial Differentiation
Extremes within restricted domains

Question
Find the maximum and minimum values of

$$f(x,y,z) = xz + yz$$

Over the ball $x^2 + y^2 + z^2 \leq 1$.
Answer
For interior critical points

$$\begin{aligned} 0 = f_1 &= z \\ 0 = f_2 &= z \\ 0 = f_3 &= x + y \end{aligned}$$

So all points on the line $z = 0$, $x + y = 0$ are critical points with $f = 0$. Consider the boundary of the ball, $x^2 + y^2 + z^2 = 1$.

$$\begin{aligned} f(x,y,z) &= (x+y)z \\ &= \pm(x+y)\sqrt{(1 + x^2 + y^2)} \\ &= g(x,y) \end{aligned}$$

With g having the domain $x^2 + y^2 \leq 1$. On the boundary of this domain, $g = 0$. However, g has both positive and negative values within the domain, so only consider CPs of g in $x^2 + y^2 < 1$

$$\begin{aligned} 0 = g_1 &= \sqrt{1 - x^x - y^2} + \frac{(x+y)(-2x)}{2\sqrt{1 - x^2 - y^2}} \\ &= \frac{1 - x^2 - y^2 - x^2 - xy}{\sqrt{1 - x^2 - y^2}} \\ 0 = g_2 &= \frac{1 - x^2 - y^2 - xy - y^2}{\sqrt{1 - x^2 - y^2}} \end{aligned}$$

$$\begin{aligned} \Rightarrow 1 &= 2x^2 + y^2 + xy \\ &= x^2 + 2y^2 + xy \\ \Rightarrow x^2 &= y^2 \end{aligned}$$

And so there are two cases:
If $x = -y$

$$\begin{aligned} g &= 0 \\ \Rightarrow f &= 0 \end{aligned}$$

If $x = y$

$$\begin{aligned} 2x^2 + x^2 + x^2 &= 1 \\ \Rightarrow x^2 &= \frac{1}{4} \\ x &= \pm\frac{1}{2} \end{aligned}$$

And so g had four CPs, with $g = \pm 1/\sqrt{2}$.

$$\begin{aligned} \min(f) &= \frac{1}{\sqrt{2}} \\ \max(f) &= -\frac{1}{\sqrt{2}} \end{aligned}$$