Applications of Partial Differentiation
Extremes within restricted domains

Question
Find the maximum and minimum values of

$$f(x, y, z) = xy^2 + yz^2$$

Over the ball $x^2 + y^2 + z^2 \leq 1$.
Answer
For interior critical points

$$\begin{aligned} 0 = f_1 &= y^2 \\ 0 = f_2 &= 2xy + z^2 \\ 0 = f_3 &= 2yz \end{aligned}$$

This makes all points on the x-axis critical points with $f = 0$
Consider the boundary of the ball, $z^2 = 1 - x^2 - y^2$.
On the boundary

$$\begin{aligned} f(x, y, z) &= xy^2 + y(1 - x^2 - y^2) \\ &= xy^2 + y - x^2y - y^3 \\ &= g(x, y) \end{aligned}$$

With g defined for $x^2 + y^2 \leq 1$.
For internal CPs of g

$$\begin{aligned} 0 = g_1 &= y^2 - 2xy + y(y - 2x) \\ 0 = g_2 &= 2xy + 1 - x^2 - 3y^2 \end{aligned}$$

$$\begin{aligned} \text{If } y = 0 & \\ \Rightarrow g &= 0 \\ f &= 0 \end{aligned}$$

$$\begin{aligned} \text{If } y = 2x & \\ \Rightarrow 0 &= 4x^2 + 1 - x^2 - 12x^2 \\ 9x^2 &= 1 \\ x &= \pm\frac{1}{3} \end{aligned}$$

This gives the critical points

$$\begin{array}{cc} \left(\frac{1}{3}, \frac{2}{3} \pm \frac{2}{3}\right) & f = \frac{12}{27} \\ \left(-\frac{1}{3}, -\frac{2}{3} \pm \frac{2}{3}\right) & f = -\frac{12}{27} \end{array}$$

Now consider $x^2 + y^2 = 1$

$$\begin{aligned} g(x, y) &= xy^2 \\ &= x(1 - x^2) \\ &= x - x^3 \\ &= h(x) \end{aligned}$$

For $-1 \leq x \leq 1$
For the end points $x = \pm 1$, $h = 0$, $\Rightarrow g = 0$ and $f = 0$.
For the critical points of h

$$\begin{aligned} 0 = h'(x) &= 1 - 3x^2 \\ \Rightarrow x &= \pm \frac{1}{\sqrt{3}} \\ y &= \pm \sqrt{\frac{2}{3}} \\ \text{with } h &= \pm \frac{2}{(3\sqrt{3})} \end{aligned}$$

But $2/(3\sqrt{3}) < 12/27$, this is not the maximum value for f

$$\begin{aligned} \Rightarrow \min(f) &= -\frac{12}{27} \\ \max(f) &= \frac{12}{27} \end{aligned}$$