Applications of Partial Differentiation
Extremes within restricted domains

Question
Find the maximum and minimum values of

$$f(x,y) = \frac{x-y}{1+x^y+y^2}$$

On the upper half-plane $y \geq 0$.
Answer
For critical points

$$\begin{aligned} 0 = f_1(x,y) &= \frac{1-x^2+y^2+2xy}{(1+x^2+y^2)^2} \\ 0 = f_2(x,y) &= \frac{-1-x^2+y^2-2xy}{(1+x^2+y^2)^2} \end{aligned}$$

With any critical points having to satisfy

$$\begin{aligned} 1-x^2+y^2+2xy &= 0 \\ -1-x^2+y^2-2xy &= 0 \\ \Rightarrow x^2 &= y^2 \\ \text{and } 2xy &= -1 \\ \Rightarrow y = -x &= \pm 1/\sqrt{2} \end{aligned}$$

And so the only critical point with $y \geq 0$ is $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, with $f = -\frac{1}{\sqrt{2}}$.
On the boundary of the plane, $y = 0$

$$f(x,0) = \frac{x}{1+x^2} = g(x), \quad (-\infty < x < \infty)$$

This gives $g(x) \to 0$ as $x \to \pm\infty$.

$$g'(x) = \frac{1-x^2}{(1+x^2)^2}$$

So the critical points of g are $x = \pm 1$, with $g(\pm 1) = \pm\frac{1}{2}$. So

$$\begin{aligned} \min(f) &= -\frac{1}{\sqrt{2}} \\ \max(f) &= \frac{1}{2}. \end{aligned}$$