Question

Show that as $x \to \infty$,

(a) $x + o(x) = O(x)$

(b) $\{O(x)\}^2 = O(x^2) = o(x^3)$

(c) $\log(1 + o(1)) = o(1)$

Answer

(a) If $f(x) = O(x)$, then

$$\lim_{x\to\infty} \frac{x + f(x)}{x} = 1 + \lim_{x\to\infty} \frac{f(x)}{x} = 1$$

Therefore $x + f(x) = O(x)$

or $x + o(x) = O(x)$.

(b) First equality:

Suppose $f(x)$ and $g(x) = O(x)$, $\quad x \to \infty$.

Then there exists K_f and $K_g > 0$ such that

$$\left.\begin{array}{l} |f(x)| \le K_f|x| \\ |g(x)| \le K_g|x| \end{array}\right\} \text{as } x \to \infty$$

Thus $|f(x)||g(x)| \le K_f K_g |x|^2$

or $|f(x)g(x)| \le K_h|x^2|$ where $K_h = K_f K_g$

$$\begin{array}{rrcl} \Rightarrow & f(x)g(x) & = & O(x^2) \\ \text{or} & [O(x^2)]^2 & = & O(x^2) \end{array}$$

Second equality:

Now suppose $\underbrace{h(x)}_{=f.g} = O(x^2) \quad x \to \infty$.

Then there exists K_h such that $|h(x)| \le K_h|x^2| \quad x \to \infty$

Thus $0 \le \lim_{x\to\infty} \left|\frac{h(x)}{x^3}\right| \le \lim_{x\to\infty} K_h \frac{|x^2|}{|x^3|} = K_h \lim_{x\to\infty} \frac{1}{|x|} = 0$

Therefore $h(x) = o(x^3)$

or $O(x^2) = o(x^3)$

Hence $[O(x)]^2 = O(x^2) = o(x^3)$

(c) Suppose $f(x) = o(1)$.

Then $\lim_{x \to \infty} \left| \frac{f(x)}{1} \right| = 0$

Thus $\lim_{x \to \infty} \log(1 + f(x)) = \log 1 = 0$

$\Rightarrow \underline{\log(1 + o(1)) = o(1)}$