Question

Find the orders of the following expressions as $\varepsilon \to 0^+$

(a) $\dfrac{1-\cos\varepsilon}{1+\cos\varepsilon}$

(b) $\dfrac{\varepsilon^{\frac{3}{2}}}{1-\cos\varepsilon}$

(c) $\text{sech}^{-1}\varepsilon$

(d) $e^{\tan\varepsilon}$

Answer

The trick is to " guess" (by expanding as $\varepsilon \to 0^+$) the answer and substitute back in to prove it.

(a) $\dfrac{1-\cos\varepsilon}{1+\cos\varepsilon} \to \dfrac{1-\left(1-\frac{\varepsilon^2}{2}+\cdots\right)}{1+\left(1+\frac{\varepsilon^2}{2}+\cdots\right)} \approx \dfrac{\varepsilon^2}{4}+\cdots$

Also $\left|\dfrac{1-\cos\varepsilon}{1+\cos\varepsilon}\right| = \dfrac{1-\cos\varepsilon}{1+\cos\varepsilon} \quad \varepsilon \to 0^+$

Therefore

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} \frac{\left|\frac{1-\cos\varepsilon}{1+\cos\varepsilon}\right|}{|\varepsilon^2|} &= \lim_{\varepsilon\to 0^+} \frac{1-\cos\varepsilon}{\varepsilon^2(1+\cos\varepsilon)} \\
&\quad \text{by l'Hôpital} \\
&= \lim_{\varepsilon\to 0^+} \frac{\sin\varepsilon}{2\varepsilon(1+\cos\varepsilon)-\varepsilon^2\sin\varepsilon} \\
&= \lim_{\varepsilon\to 0^+} \frac{\cos\varepsilon}{2(1+\cos\varepsilon)-4\varepsilon\sin\varepsilon-\varepsilon^2\cos\varepsilon} \\
&= \frac{1}{4} > 0
\end{aligned}$$

$$\Rightarrow \frac{1-\cos\varepsilon}{1+\cos\varepsilon} = O(\varepsilon^2) \quad \varepsilon \to 0^+ \qquad \left(k \geq \frac{1}{4}\right)$$

(b) $\dfrac{\varepsilon^{\frac{3}{2}}}{1-\cos\varepsilon} \to \dfrac{\varepsilon^{\frac{3}{2}}}{1-1+\frac{\varepsilon^2}{2}+\cdots} \approx 2\dfrac{\varepsilon^{\frac{3}{2}}}{\varepsilon^2} = \dfrac{2}{\varepsilon^{\frac{1}{2}}} \quad \varepsilon \to 0^+$

Also $\left|\dfrac{\varepsilon^{\frac{3}{2}}}{1-\cos\varepsilon}\right| = \dfrac{\varepsilon^{\frac{3}{2}}}{1-\cos\varepsilon} \quad \varepsilon \to 0^+$

Therefore

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \frac{\left|\frac{\varepsilon^{\frac{3}{2}}}{1-\cos\varepsilon}\right|}{\left|\frac{1}{\varepsilon^{\frac{1}{2}}}\right|} &= \lim_{\varepsilon\to 0^+} \frac{\varepsilon^2}{1-\cos\varepsilon} \\ &= \lim_{\varepsilon\to 0^+} \frac{2\varepsilon}{\sin\varepsilon} \\ &= \lim_{\varepsilon\to 0^+} \frac{2}{\cos\varepsilon} \\ &= 2 > 0\end{aligned}$$

$\Rightarrow \dfrac{\varepsilon^{\frac{3}{2}}}{1-\cos\varepsilon} = O(\varepsilon^{-\frac{1}{2}}) \;\; \varepsilon \to 0+ \quad (k \geq 2)$

(c) Let $\text{sech}^{-1}\varepsilon = u$ so that $\text{sech} u = \varepsilon \;\Rightarrow \cosh u = \frac{1}{\varepsilon}$

PICTURE (both graphs)

$\Rightarrow e^u + e^{-u} = \dfrac{2}{\varepsilon}$. Now as $\varepsilon \to 0,\; u \to \infty$ so we have

$e^u \approx \dfrac{2}{\varepsilon}$ or $u \approx \ln 2 + \ln\left(\frac{1}{\varepsilon}\right) \Rightarrow \text{sech}^{-1}\varepsilon \approx \ln 2 + \ln\left(\frac{1}{\varepsilon}\right) \quad \varepsilon \to 0$

So try $\text{sech}^{-1}\varepsilon = O\left(\ln\left(\dfrac{1}{\varepsilon}\right)\right)$:

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \left|\frac{\text{sech}^{-1}\varepsilon}{\ln(\frac{1}{\varepsilon})}\right| &= \lim_{\varepsilon\to 0^+} \frac{\text{sech}^{-1}\varepsilon}{\ln(\frac{1}{\varepsilon})} \\ &= \lim_{\varepsilon\to 0^+} \frac{1}{-\frac{1}{\varepsilon}} \times -\frac{1}{\varepsilon\sqrt{1-\varepsilon^2}} \\ &= \lim_{\varepsilon\to 0^+} \frac{1}{\sqrt{1-\varepsilon^2}} \\ &= 1 > 0\end{aligned}$$

$\Rightarrow \text{sech}^{-1}\varepsilon = O\left(\ln\left(\dfrac{1}{\varepsilon}\right)\right) \quad$ as $\varepsilon \to 0^+$

(d) $e^{\tan\varepsilon} \approx e^{\varepsilon+\cdots} \approx 1+\varepsilon+\cdots$ as $\varepsilon \to 0^+$

Therefore try $O(1)$.

$|e^{\tan\varepsilon}| = e^{\tan\varepsilon}$ as $\varepsilon \to 0^+$

Therefore $\lim_{\varepsilon\to 0^+} \left|\frac{e^{\tan\varepsilon}}{1}\right| = \lim_{\varepsilon\to 0^+} e^{\tan\varepsilon} = 1$ (regular limit)

$\Rightarrow e^{\tan\varepsilon} = O(1),\ \varepsilon \to 0^+ \quad (k \geq 1)$