Question
For small real positive ε show that

(a) $\sinh\left(\frac{1}{\varepsilon}\right) = O\left(e^{\frac{1}{\varepsilon}}\right)$

(b) $\log(1+\sin\varepsilon) = O(\varepsilon)$

(c) $\log(2+\sin\varepsilon) = O(1)$

Answer

(a) $\sinh\left(\frac{1}{\varepsilon}\right) = \frac{\mathrm{e}^{\frac{1}{\varepsilon}} - \mathrm{e}^{-\frac{1}{\varepsilon}}}{2}$.

Therefore

$$\lim_{\varepsilon\to 0^+} \left| \frac{\sinh\left(\frac{1}{\varepsilon}\right)}{e^{\frac{1}{\varepsilon}}} \right| = \lim_{\varepsilon\to 0} \frac{e^{\frac{1}{\varepsilon}} - e^{-\frac{1}{\varepsilon}}}{2e^{\frac{1}{\varepsilon}}} = \lim_{\varepsilon\to 0} \frac{1}{2} - \frac{1}{2}e^{-\frac{2}{\varepsilon}} \frac{1}{2} > 0$$

$$\Rightarrow \sinh\left(\frac{1}{\varepsilon}\right) = O\left(\frac{1}{\varepsilon}\right),\ \varepsilon \to 0^+$$

(b) $\log(1+\sin\varepsilon) > 0$ as $\varepsilon \to 0^+$

Therefore

$$\begin{aligned} \lim_{\varepsilon\to 0^+} \left| \frac{\log(1+\sin\varepsilon)}{\varepsilon} \right| &= \lim_{\varepsilon\to 0^+} \left[\frac{\log(1+\sin\varepsilon)}{\varepsilon} \right] \\ &\quad \text{by l'Hôpital} \\ &= \lim_{\varepsilon\to 0^+} \frac{\cos\varepsilon}{1+\sin\varepsilon} = 1 > 0 \end{aligned}$$

$$\Rightarrow \log(1+\sin\varepsilon) = O(\varepsilon) \quad \varepsilon \to 0^+$$

(c) $\log(2+\sin\varepsilon) > 0$ as $\varepsilon \to 0^+$

Therefore

$$\begin{aligned} \lim_{\varepsilon\to 0^+} \left| \frac{\log(2+\sin\varepsilon)}{1} \right| &= \lim_{\varepsilon\to 0^+} \left[\frac{\log(2+\sin\varepsilon)}{1} \right] \\ &= \log 2 > 0 \end{aligned}$$

$$\Rightarrow \log(2+\sin\varepsilon) = O(1) \text{ as } \varepsilon \to 0^+$$