Question

For $|z| \to \infty$ find the sectors in the complex plane where the following order estimates are satisfied for all positive n.

(a) $z^n = o(e^z)$

(b) $e^z = o(z^n)$

(c) $z^n = o(e^{z^2})$

Answer

(a) $z^n = o(e^z)$ as $|z| \to \infty \Rightarrow \lim\limits_{|z|\to\infty} \left|\frac{z^n}{e^z}\right| = 0$, for all $n > 0$.

This is true for $Re(z) > 0,\ (|e^z| \gg 1)$

$\Rightarrow z^n = o(e^z),\ |z| \to \infty,\ Re(z) > 0$

(b) $e^z = o(z^n) \Rightarrow \lim_{|z|\to\infty} \left|\frac{e^z}{z^n}\right| = 0$ for all $n > 0$

When $Re(z) < 0, \quad |e^z| \ll 1$.

Therefore $e^z = o(z^n),\ |z| \to \infty,\ Re(z) < 0$

(c) $z^n = o(e^{z^2}) \Rightarrow \lim\limits_{|z|\to\infty} \left|\frac{z^n}{e^{z^2}}\right| = 0$ for all $n > 0$

Here $|e^{z^2}| > |z^n|$ for all $z \to \infty$.

Therefore $z^n = o(e^{z^2}),\ z \to \infty$.

except on the imaginary axis, when $|e^{z^2}| = 1 \quad Re(z) \neq 0$.