**Question**

Use the WKB solution to estimate the eigenvalues $k$ of the problem

$$y'' + \frac{k^2}{x^2}y = 0,\ y(1) = 0,\ y(e) = 0,\ k \to +\infty$$

Show that the equation can be solved exactly to give

$$y = a\sqrt{x}\cos\left(\sqrt{k^2 - \frac{1}{4}}\log x\right) + b\sqrt{x}\sin l\left(\sqrt{k^2 - \frac{1}{4}}\log x\right)$$

Hence find the exact eigenvalues and compare them with the WKB approximations. Will higher order terms in the WKB expansion improve the accuracy of that approximation?

**Answer**

$y'' + \frac{k^2}{x^2}y = 0,\ \underbrace{y(1) = 0,\ y(e) = 0}_{\Rightarrow x = O(1),\ k \to \infty \text{ limit}}$

Can solve by $y \sim e^{g_0(k)\psi_0(x)+\cdots}$, $k \to \infty$ limit ansatz, or jump to equations (5.61) (to save space here!) in section 5.5 of the notes. Identify $h(x) = \frac{1}{x^2}$ with $h(x) > 0$ in $1 \leq x \leq e$.

Therefore from (5.61) we have

$y \sim \frac{A}{(\frac{1}{x^2})^{\frac{1}{4}}}\cos\left\{k\int^x \sqrt{\frac{1}{x^2}}\,dx\right\} + \frac{B}{(\frac{1}{x^2})^{\frac{1}{4}}}\sin\left\{k\int^x \sqrt{\frac{1}{x^2}}\,dx\right\},$

$k \to \infty,\ x = O(1)$

so

$y \sim A\sqrt{x}\cos(k\ln x) + B\sqrt{x}\sin(k\ln x),\ k \to \infty,\ x = O(1)$

$A$ and $B$ from boundary conditions

$\Rightarrow \begin{array}{l} 0 \sim A\sqrt{1}\cos(k\ln 1) + B\sqrt{1}\sin(k\ln 1) \\ 0 \sim A\cos 0 + B\sin 0 \end{array}$

$\Rightarrow \underline{A = 0}$

and

$0 \sim A\sqrt{e}\cos(k\ln e) + B\sqrt{e}\sin(k\ln e)$

$0 \sim B\sqrt{e}\sin k$ since $A = 0$. $\ln e = 1$ by definition

so either $B = 0$ BORING! ($y \sim 0$) or $\sin k = 0 \Rightarrow k = n\pi,\quad n \in \mathbf{Z}$

Therefore $y \sim B\sqrt{x}\sin(n\pi\ln x)$ with eigenvalues $\underline{k_n \sim \pi}, n$ integer.

Given trial solution:

$$
\begin{aligned}
y &= a\sqrt{x}\cos\left(\sqrt{k^2-\frac{1}{4}}\log x\right)+b\sqrt{x}\sin\left(\sqrt{k^2-\frac{1}{4}}\log x\right)\\
y' &= \frac{1}{2}ax^{-\frac{1}{2}}\cos\left(\sqrt{k^2-\frac{1}{4}}\log x\right)+bx^{-\frac{1}{2}}\sin\left(\sqrt{k^2-\frac{1}{4}}\log x\right)\\
&\quad -a\frac{\sqrt{x}\sqrt{k^2-\frac{1}{4}}}{x}\sin\left(\sqrt{k^2-\frac{1}{4}}\log x\right)\\
&\quad +\frac{b\sqrt{x}\sqrt{k^2-\frac{1}{4}}}{x}\cos\left(\sqrt{k^2-\frac{1}{4}}\log x\right)\\
&= \left(\frac{a}{2}+b\sqrt{k^2-\frac{1}{4}}\right)\frac{\cos\left(\sqrt{k^2-\frac{1}{4}}\log x\right)}{\sqrt{x}}\\
&\quad +\left(\frac{b}{2}-a\sqrt{k^2df14}\right)\frac{\sin\left(\sqrt{k^2-\frac{1}{4}}\log x\right)}{\sqrt{x}}\\
y'' &= -\frac{(\frac{a}{2}+b\sqrt{k^2-\frac{1}{4}})\sqrt{k^2-\frac{1}{4}}\sin\left(\sqrt{k^2-\frac{1}{4}}\log x\right)}{x^{\frac{3}{2}}}\\
&\quad +\frac{(\frac{b}{2}-a\sqrt{k^2-\frac{1}{4}})\sqrt{k^2-\frac{1}{4}}\sin\left(\sqrt{k^2-\frac{1}{4}}\log x\right)}{x^{\frac{3}{2}}}\\
&\quad -\frac{\frac{1}{2}(\frac{a}{2}+b\sqrt{k^2-\frac{1}{4}})\cos\left(\sqrt{k^2-\frac{1}{4}}\log x\right)}{x^{\frac{3}{2}}}\\
&\quad -\frac{\frac{1}{2}(\frac{a}{2}+b\sqrt{k^2-\frac{1}{4}})\sin\left(\sqrt{k^2-\frac{1}{4}}\log x\right)}{x^{\frac{3}{2}}}\\
&= \left(-b\left[k^2-\frac{1}{4}\right]-\frac{b}{4}\right)\frac{\sin\left(\sqrt{k^2-\frac{1}{4}}\log x\right)}{x^{\frac{3}{2}}}\\
&\quad +\left(-a\left[k^2-\frac{1}{4}\right]-\frac{a}{4}\right)\frac{\cos\left(\sqrt{k^2-\frac{1}{4}}\log x\right)}{x^{\frac{3}{2}}}\\
&= -bk^2\frac{\sin\left(-b\left[k^2-\frac{1}{4}\right]-\frac{b}{4}\right)\frac{\sin\left(\sqrt{k^2-\frac{1}{4}}\log x\right)}{x^{\frac{3}{2}}}}{x^{\frac{3}{2}}}\\
&\quad -ak^2\frac{\cos\left(-b\left[k^2-\frac{1}{4}\right]-\frac{b}{4}\right)\frac{\sin\left(\sqrt{k^2-\frac{1}{4}}\log x\right)}{x^{\frac{3}{2}}}}{x^{\frac{3}{2}}}\\
&= \frac{-k^2}{x}y\surd\surd
\end{aligned}
$$

so it is an exact solution of equation.

Moreover, it satisfies the boundary conditions provided.

$$
\begin{aligned}
0 &= a\sqrt{1}\cos\left(\sqrt{k^2-\frac{1}{4}}\log 1\right)+b\sqrt{1}\sin\left(\sqrt{k^2-\frac{1}{4}}\log 1\right)\\
\Rightarrow 0 &= a\\
\text{and } 0 &= b\sqrt{e}\sin\left(\sqrt{k^2-\frac{1}{4}}\log e\right)\\
\Rightarrow 0 &= b\sin\left(\sqrt{k^2-\frac{1}{4}}\right)
\end{aligned}
$$

Therefore either $b = 0$ BORING! $y = 0$ for all $x$

$$
k^2 = n^2\pi^2 + \frac{1}{4}
$$

or $\sqrt{k^2-\frac{1}{4}} = n\pi \Rightarrow$

$$
\begin{aligned}
k &= \pm\sqrt{n^2\pi^2+\frac{1}{4}}\\
&= \pm n\pi\sqrt{1+\frac{1}{4n\pi}}
\end{aligned}
$$

Therefore boundary data $\Rightarrow$ eigenvalues are

$$
k_n = \pm n\pi\left(1+\frac{1}{8n\pi}+O\left(\frac{1}{n^2}\right)\right) \text{ as n} \to \infty
$$

This agrees for $n \to \pm\infty$ with the WKB result $k_n \sim n\pi$ as above.
Next WKB term (long calculation) gives evalues from

$$
y \sim a\sqrt{x}\cos\left(\left(k-\frac{1}{8k}\right)\log x\right)+b\sqrt{x}\sin\left(\left(k-\frac{1}{8k}\right)\log x\right)
$$

which when substituted into boundary conditions

$$
\begin{aligned}
\Rightarrow k-\frac{1}{8k} = n\pi \quad &\Rightarrow \quad 8k^2-8kn\pi-1=0\\
&\Rightarrow \quad k = \frac{8n\pi \pm \sqrt{64n^2\pi^2+32}}{16}\\
&\Rightarrow \quad k \sim n\pi + \frac{1}{8n\pi}+o\left(\frac{1}{n^2}\right),\ n\to\infty
\end{aligned}
$$

which agrees with exact?