Question

Find WKB solutions for $x = O(1),\ k \to +\infty$

$$y'' + \left(k + \frac{1}{2} - \frac{1}{4}x^2\right) y = 0$$

Answer

$y'' + \left(k + \frac{1}{2} - \frac{1}{4}x^2\right) y = 0,\ x = O(1),\ k \to +\infty$

Try ansatz

$y \sim \exp\{g_0(k)\psi_0(x) + g_1(k)\psi_1(k) + \cdots\}$

$\{g_r(k)\}$ form asmyptotic sequence as $k \to \infty$

$\{\psi_r(x)\} = O(1)$ for $x = O(1),\ k \to +\infty$

$y' \sim (g_0\psi_0' + g_1\psi_1' + \cdots)\exp\{g_0\psi_0 + g_1\psi_1 + \cdots\}$

$y'' \sim ((g_0\psi_0'' + g_1\psi_1'' + \cdots) + (g_0\psi_0' + g_1\psi_1' + \cdots)^2) \times \exp\{g_0\psi_0 + g_1\psi_1 + \cdots\}$

Substitute into equation and simplify

$(g_0\psi_0'' + g_1\psi_1'' + \cdots) + (g_0\psi_0' + g_1\psi_1' + \cdots)^2 + k + \frac{1}{2} - \frac{1}{4}x^2 = 0$

Now by asymptotic sequence property, we can assume that dominant behaviour is given by

$$g_0\psi_0'' + \underbrace{g_0^2{\psi_0'}^2 + 2g_0g_1\phi_0'\psi_1'} + k + \frac{1}{2} - \frac{1}{4}x^2 = 0$$

$$g_0g_1 = o(g_0^2)$$

so

$$\underbrace{g_0\psi_0'' + g_0^2{\psi_0'}^2} + \underbrace{k + \frac{1}{2} - \frac{1}{4}}x^2 = 0$$

assume $g_0 = o(g_0^2)$ $\quad x = o(k),\ 1 = o(k)$ as $k \to \infty\ \ x = O(1)$

so

$g_0^2{\psi_0'}^2 = -k \Rightarrow g_0 = \sqrt{k},$

${\psi_0'}^2 = -1$ put $\pm\sqrt{}$ ambiguities into ψ_0 for convenience

$\psi_0' = \pm i$

$\psi_{\pm}ix + const$ absorb the constant into the exponential prefactor

Thus we return to the above equation:

$$\sqrt{k} \cdot 0 - k + 2\sqrt{k}g_1(\pm i)\psi_1' + k + \frac{1}{2} - \frac{1}{4}x^2 = 0$$

$$\pm 2i\sqrt{kg_1}\psi_1' = -\frac{1}{2} + \frac{1}{4}x^2$$

To obtain balance at $O(k^0)$ we therefore need

$$
g_1 = k^{-\frac{1}{2}} \Rightarrow \quad
\begin{aligned}
\pm 2i\psi_1' &= -\frac{1}{2} + \frac{1}{4}x^2 \\
\psi_1' &= \frac{1}{\pm 2i}\left(-\frac{1}{2} + \frac{1}{4}x^2\right) \\
\psi_1 &= \frac{1}{2 \pm 2i}\left(-\frac{x}{2} + \frac{x^3}{12}\right) \\
&= pm\frac{i}{4}\left(x - \frac{x^3}{6}\right)
\end{aligned}
$$

Next order balance is given by:

$$
(\overbrace{g_0\psi_0''}^{=0} + g_1\psi_1'' + \cdots) + (2g_0g_1\psi_0'\psi_1' + 2g_0g_2\psi_0'\psi_2' + \cdots)
$$
$$
+(g_0^2\psi_0^{2\prime} + g_1^2\psi_1^{2\prime} + \cdots) + k + \frac{1}{2} - \frac{1}{4}x^2 = 0
$$
$$
0 = k^{-frac12}\psi_1'' + 2\psi_0'\psi_1' + 2k^{\frac{1}{2}}g_2\psi_0'\psi_2' + k\psi_0^{2\prime} + \underbrace{\frac{\psi_1^{2\prime}}{k}}_{\text{must be negligible at } O(k^{-\frac{1}{2}})} + k + \frac{1}{2} - \frac{1}{4}x^2
$$

Balance at $O(k^{-\frac{1}{2}})$ only if

$0 = k^{-\frac{1}{2}}\psi_1'' + 2k^{\frac{1}{2}}g_2\psi_0'\psi_2'$

$\Rightarrow g_2 = \frac{1}{k}$ and $0 = \mp\frac{i}{4}x \pm 2i\psi_2'$

$\Rightarrow \psi_2' = \frac{x}{8} \Rightarrow \psi_2 = \frac{x^2}{16} + const$ absorb the constant into the exponential prefactor

etc...

Drawing this together we see:

$$
\begin{aligned}
Y \quad \sim \quad & A\exp\left\{+ix\sqrt{k} + \frac{i}{4}\left(x - \frac{x^3}{6}\right)k^{-\frac{1}{2}} + \frac{x^2}{16k} + \cdots\right\} \\
& +B\exp\left\{-ix\sqrt{k} - \frac{i}{4}\left(x - \frac{x^3}{6}\right)k^{-\frac{1}{2}} + \frac{x^2}{16k} + \cdots\right\}
\end{aligned}
$$