Question

Find two terms in the asymptotic solutions of the equation

$$y'' - \frac{k^4(x^2+1)}{k^2+x^2}y = 0,$$

(a) for $x \to +\infty,\ k = O(1)$

(b) for $k \to +\infty,\ x = O(1)$

Answer

$y'' - \dfrac{k^4(x^2+1)}{k^2+x^2}y = 0$

(a) $\underline{x \to +\infty,\ k = O(1)}$:

Try $y \sim e^{\psi_0(x)+\psi_1(x)}$

$\phi_r\} =$ asymptotic sequence as $x \to +\infty$

Substitute:

$\Rightarrow (\phi_0''+\phi_1''+\cdots)+({\phi_0'}^2+{\phi_1'}^2+\cdots)+(2\phi_0'\phi_1'+2\phi_0'\phi_2'+\cdots)-\dfrac{k^4(x^2+1)}{k^2+x^2} \sim 0, \quad x \to +\infty$

Now as $x \to +\infty,\ k = O(1)$

$$\begin{aligned}
\frac{k^4(x^2+1)}{k^2+x^2} &\sim k^4\left(1+\frac{1}{x^2}\right)\left(1+\frac{k^2}{x^2}\right)^{-1} \\
&\sim k^4\left(1+\frac{1}{x^2}\right)\left(1-\frac{k^2}{x^2}+O\left(\frac{1}{x^4}\right)\right) \\
&\sim k^4\left(1+\frac{(1-k^2)}{x^2}+O\left(\frac{1}{x^4}\right)\right)
\end{aligned}$$

Assuming that $\{\phi_r''\}$ and $\{\phi_r'\}$ are asymptotic sequences also, the first possible balance is

$$\phi_0'' + {\phi_0'}^2 - k^4 \sim 0$$

Checking through pairwise balances, the only sensible balance is

${\phi_0'}^2 = k^4 \Rightarrow \phi_0 \pm k^2,\ \phi_0 = \pm k^2 x + const$

Absorb the constant into the exponential prefactor

$(\phi_0'' = o(k^4)\surd\surd)$

Next balance

$$(0 + \phi_1'' + \cdots) +)k^4 + {\phi_1'}^2 + \cdots) + (2\phi_0\phi_1' + \cdots) - k^4 - \frac{k^4(1-k^2)}{x^2} \sim 0$$

${\phi_1'}^2 = o(\phi_0'\phi_1')$ by a symmetric sequence property

Therefore $\phi_1'' + 2\phi_0'\phi_1' - \dfrac{k^4(1-k^2)}{x^2} \sim 0$ is the next possible balance.

Checking through pairwise, the only sensible balance is

$$\begin{aligned} \pm 2k^2\phi_1' &= \frac{k^4(1-k^2)}{x^2} \\ \Rightarrow \phi_1' &= \pm\frac{k^2(1-k^2)}{2x^2} \Rightarrow \phi_1'' = O\left(\frac{1}{x^3}\right) = o\left(\frac{1}{x^2}\right) \surd\surd \\ \phi_1 &= \mp\frac{k^2(1-k^2)}{2x} \end{aligned}$$

Therefore $y \sim A_+ e^{+k^2x - \frac{k^2(1-k^2)}{2x}} + A_- e^{-k^2x + \frac{k^2(1-k^2)}{2x}}$

(b) $\underline{k \to +\infty,\ x = O(1)}$:

Try $y \sim e^{g_0(k)\psi_0(x) + g_1(k)\psi_1(x) + \cdots}$

$\{g_r\}$ asymptotic sequence as $k \to \infty,\ x = O(1),\ \{\psi + r\} = O(1)$

Substitute

$$(g_0\psi_0'' + g_1\psi_1'' + \cdots) + (g_0^2{\psi_0'}^2 + g_1^2{\psi_1'}^2 + \cdots) + (2g_0g_1\psi_0'\psi_1' + 2g_0g_2\psi_0'\psi_2' + \cdots) - \frac{k^4(x^2+1)}{k^2+x^2} = 0$$

As $k \to +\infty,\ x = O(1)$

$$\begin{aligned} \frac{k^4(x^2+1)}{k^2+x^2} &\sim \frac{k^4(x^2+1)}{k^2}\left(1 + \frac{x^2}{k^2}\right)^{-1} \\ &\sim k^2(x^2+1)\left(1 - \frac{x^2}{k^2} + O\left(\frac{1}{k^4}\right)\right) \\ &\sim k^2(x^2+1) - x^2(x^2+1) + O\left(\frac{1}{k^2}\right) \end{aligned}$$

<u>Balance at $O(k^2)$</u> (is most denominators as $k \to +\infty$)

$g_0^2{\psi_0'}^2 - k^2(x^2+1) \sim 0$

$\Rightarrow g_0 = k$ and ${\psi_0'}^2 = x^2 + 1$

$\psi_0' = \pm\sqrt{x^2+1} \Rightarrow \pi_0 = \pm\dfrac{1}{2}[\mathrm{arcsinhx + x\sqrt{1+x^2}}]$

Next balance at $O(k)$:

$(k\psi_0'' + \cdots) + (k^2{\psi_0'}^2 + g_1^2{\psi_1'}^2 + \cdots) + (2kg_1\psi_0'\psi_1' + \cdots) - k^2(x^2+1) + x^2\underbrace{(x^2+1)}_{} \sim 0$

$= O(k^0)$ Therefore neglect

For balance must have:

$$k\psi_0'' + 2kg_1\psi_0'\psi_1' \sim 0$$

$$\Rightarrow g_1 = 1 (= k^0) \text{ and } \psi_0'' + 2\psi_0'\psi_1' \sim 0$$

$$\Rightarrow \psi_1' = \frac{-\psi_0''}{2\psi_0'} = \mp\frac{1}{2}\frac{\frac{1}{2}(x^2+1)^{-\frac{1}{2}}}{\sqrt{x^2+1}} \cdot 2x = \mp\frac{x}{2(x^2+1)}$$

$$\psi_1 = \mp\frac{1}{4}\ln(x^2+1)$$

Therefore

$$\begin{aligned} y \sim\ & \frac{A_+}{(x^2+1)^{\frac{1}{4}}}\exp\left\{\frac{1}{2}\text{arcsinh x} + \frac{\text{x}}{2}\sqrt{1+\text{x}^2}\right\} \\ & + A_-(x^2+1)^{\frac{1}{4}}\exp\left\{-\frac{1}{2}\text{arcsinh x} - \frac{\text{x}}{2}\sqrt{1+\text{x}^2}\right\} \end{aligned}$$