Question
Find two terms in the asmyptotic solutions of the equation

$$y'' + k^2\left(\frac{x^2+2}{x^2+1}\right)^2 y = 0,$$

(a) for $x \to +\infty,\ k = O(1)$

(b) for $k \to +\infty,\ x = O(1)$

Answer
$y'' + k^2\left(\frac{x^2+2}{x^2+1}\right)^2 y = 0$

(a) $\underline{x \to +\infty,\ k = O(1)}$: Try $y \sim e$

$\{\phi_r\}$ = asymptotic sequence as $x \to +\infty$

Substitute:

$\Rightarrow\ (\phi_0'' + \phi_1'' + cdots) + (\phi_0^{2\prime} + \phi_1^{2\prime} + \cdots) + (2\phi_0'\phi_1' + 2\phi_0'\phi_2' + \cdots) + k^2\left(\frac{x^2+2}{x^2+1}\right)^2 = 0$

Now as $x \to \infty,\ k = O(1)$

$$\begin{aligned} k^2\left(\frac{x^2+2}{x^2+1}\right)^2 &\sim k^2\left(1+\frac{2}{x^2}\right)^2\left(1+\frac{1}{x^2}\right)^{-2} \\ &\sim k^2\left(1+\frac{4}{x^2}+\frac{4}{x^2}\right)\left(1-\frac{2}{x^2}+O\left(\frac{1}{x^4}\right)\right) \\ &\sim k^2\left(1+\frac{2}{x^2}+O\left(\frac{1}{x^4}\right)\right) \end{aligned}$$

Assuming $\{\phi_r''\}$ and $\{\phi_r'\}$ are asymptotic sequences also,

$\phi_0''+\phi_0^{2\prime}+k^2 \sim 0$ is the first possible balance as $x \to +\infty$ $(\phi_0'\phi_1' = o({\phi_0'}^2))$ etc...

Checking through pairwise balances, the only sensible balance is:

${\phi_0'}^2 = -k^2 \Rightarrow \underline{\phi_0 = \pm ikx},\ (\phi_0'' = 0 = o(k))\surd\surd$

Next balance

$$(0 + \phi_1'' + \cdots) + (-k^2 + {\phi_1'}^2 + \cdots + (2\phi_0'\phi_1' + \cdots) + k^2 + \frac{2k^2}{x^2} \sim 0$$

${\phi_1'}^1 = o(\phi_0'\phi_1')$ by asymptotic sequence property

Therefore $\phi_1'' + 2\phi_0'\phi_1' + \dfrac{2k^2}{x^2} \sim 0$ is next possible balance

Checking through pairwise balances, the only sensible balance is

$$\pm 2i\phi_1' k \sim \frac{-2k^2}{x^2} \begin{array}{rcl} \Rightarrow \;\; \phi_1' & = & \pm \dfrac{ik}{x^2} \quad \left(\phi_1'' = O\left(\dfrac{1}{x^3}\right) = o\left(\dfrac{1}{x^2}\right)\right) \\ \Rightarrow \;\; \phi_1 & = & \mp \dfrac{ik}{x} + const \end{array}$$

absorb the constant in the exponential prefactor

Therefore

$y \sim A_\pm e^{\pm ikx \mp \frac{ik}{x}} = A_+ e^{ik(x-\frac{1}{x})} + A_- e^{-ik(x-\frac{1}{x})}$

(b) $\underline{k \to +\infty,\ x = O(1)}$:

Try $y \sim e^{g_0(k)\psi_0(x) + g_1(k)\psi_1(x) + \cdots}$

$\{g_r\}$ asymptotic sequence as $k \to +\infty, \{\psi_r\} = O(1)$

Substitute

$(g_0\psi_0'' + g_1\psi_1'' + \cdots) + (g_0^2{\psi_0'}^2 + g_1^2{\psi_1'}^2 + \cdots) + (2g_0g_1\psi_0'\psi_1' + 2g_0g_2\psi_0'\psi_2' + \cdots + 2g_1g_2\psi_1'\psi_2' + \cdots) + k^2\left(\dfrac{x^2+2}{x^2+1}\right)^2 = 0$

<u>Balance at $O(k^2)$</u>

$$g_0^2{\psi_0'}^2 + k^2\left(\frac{x^2+2}{x^2+1}\right)^2 = 0$$

$\Rightarrow g_0 = k$ and ${\psi_0'}^2 = -\left(\dfrac{x^2+2}{x^2+1}\right)^2$

Therefore $\psi_0' = \pm i\left(\dfrac{x^2+2}{x^2+1}\right)$,

$\psi_0 = \pm i \displaystyle\int^x \left(\frac{1}{1+x^2} + 1\right) dx = \pm i\,(x + \arctan x) + c$

absorb c into the exponential prefactor

<u>Balance at $O(k)$</u> is next

$(k\psi_0'' + \cdots) + (k^2{\psi_0'}^2 + g_1^2{\psi_1'}^2 \cdots) + (2kg_1\psi_0'\psi_1' + \cdots) + k^2\left(\dfrac{x^2+2}{x^2+1}\right)^2 \sim 0$

For balance must have

$$k\psi_0'' + 2kg_1\psi_0'\psi_1' \sim 0$$

$\Rightarrow g_1 = 1 (= k^0)$ and $\psi_0'' + 2\psi_0'\psi_1' \sim 0$

$$\Rightarrow \psi_1' = \frac{-\psi_0''}{2\psi_0'} = -\frac{1}{2}\frac{\mp i2x}{\pm i(\frac{x^2+2}{x^2+1})(x^2+1)^2}$$

$$= +\frac{x}{(x^2+2)(x^2+1)}$$

$$= \frac{x}{x^2+1} - \frac{x}{x^2+2}$$

$$\Rightarrow \psi_1 = \frac{1}{2}\ln(x^2+1) - \frac{1}{2}\ln(x^2+2) + const$$

absorb constant into exponential prefactor

Therefore $y \sim A_\pm \exp\left\{\pm ki(x + \arctan x) + \ln\sqrt{\frac{x^2+1}{x^2+2}}\right\}$

Therefore

$$y \sim A_+\sqrt{\frac{x^2+1}{x^2+2}}\exp\{+ik[x + \arctan x]\}$$
$$A_-\sqrt{\frac{x^2+1}{x^2+2}}\exp\{-ik[x + \arctan x]\}$$

<u>or</u>

$$y \sim B_+\sqrt{\frac{x^2+1}{x^2+2}}\cos[k(x + \arctan x)]$$

$$+B_-\sqrt{\frac{x^2+1}{x^2+2}}\cos[k(x + \arctan x)]$$