Question
Let

$$y'' - x^a y = 0$$
$$y \sim \exp\{\phi_0(x) + \phi_1(x) + \phi_2(x) + \cdots\}$$

where $\{\phi_n(z)\}$ is an asymptotic sequence as $x \to +\infty$. Show that the dominant terms are

$$\begin{array}{ll} \phi_0^{2\prime} - x^a \sim 0, & a > -2 \\ \phi_0{}'' + \phi_0^{2\prime} - x^a \sim 0, & a = -2 \\ \phi_0'' - x^a \sim 0, & a < -2 \end{array}$$

Hence find the leading terms in the asymptotic expansions in each case.

Answer
$y'' - x^a y = 0,\ y \sim e^{\{\phi_0 + \phi_1 + \phi_2 + \cdots\}},\ \{\phi_r\}$ asymptotic sequence as $r \to \infty$.
Substitute to get:
$(\phi_0'' + \phi_1'' + \phi_2'' + \cdots) + ({\phi_0'}^2 + {\phi_1'}^2 + {\phi_2'}^2 + \cdots)$
$$+\begin{pmatrix} 2\phi_0'\phi_1' & + & 2\phi_0'\phi_2' & + & \cdots \\ & + & 2\phi_1'\phi_2' & + & \cdots \\ & & & + & \cdots \end{pmatrix} = x^a$$
$(\phi_0'\phi_r' = o({\phi_0'}^2)$ etc. $\phi_1'\phi_r' = o(\phi_0'\phi_r')$ etc.)
Focus on this as dominant balance.

1. $\underline{\text{If } \phi_0'' \sim \mathrm{x^a}}$ $(x \to +\infty) \Rightarrow \phi_0' \sim \dfrac{x^{a+1}}{a+1},\ \phi_0 \sim \dfrac{x^{a+2}}{(a+1)(a+2)}$

   Consider size of ${\phi_0'}^2 = O(x^{2a+2})$

   For this to be consistent we therefore need

   $x^{2a+2} = o(\phi_0'' \text{ or } \mathrm{x^a}) = \mathrm{o(x^a)}$

   Therefore $2a + 2 < 2$ as $\mathrm{x} \to +\infty$

   $\Rightarrow \underline{a < -2}$

2. $\underline{If\ {\phi_0'}^2 \sim x^a}$ $(x \to +\infty) \Rightarrow \begin{array}{l} \phi_0' \sim \pm x^{\frac{a}{2}} \\ \phi_0 \sim \dfrac{\pm x^{\frac{a}{2}+1}}{(\frac{a}{2}+1)} \end{array}$

   For this to be consistent we need

$$\phi - 0'' = o(x^a)$$

$$\Rightarrow \frac{\pm(\frac{a}{2}+1)(\frac{a}{2})}{(\frac{a}{2}+1)} x^{\frac{a}{2}-1} = o(x^a)$$
$$\Rightarrow \frac{a}{2} - 1 < a \quad x \to +\infty$$
$$\Rightarrow \underline{a > -2}$$

3. If all three are the same size,

$$\phi_0'' + {\phi_0'}^2 = x^a$$

Suppose $\phi_0' = \alpha x^b$, say, $\alpha,\ \beta$ consts

$$\Rightarrow \phi_0'' = \alpha b x^{b-1}$$

Therefore need $\alpha b x^{b-1} + \alpha^2 x^{2b} = x^a \quad x \to +\infty$
or $b - 1 = 2b = a$ balancing powers $\Rightarrow b = -1,\ a = -2$
Also need $\alpha v + \alpha^2 = 1 \Rightarrow \alpha^2 - \alpha - 1 = 0$
$\alpha = \frac{1 \pm \sqrt{5}}{2}$
$\Rightarrow \phi_0 = \left(\frac{1 \pm \sqrt{5}}{2}\right) \log x$

So summary:

$$\left.\begin{array}{lll} (i)\ {\phi_0'}^2 \sim x^a & \text{if} & a > -2 \\ (ii)\ \phi_0'' + {\phi_0'}^2 \sim x^a & \text{if} & a = -2 \\ (iii)\ \phi_0'' \sim z^a & \text{if} & a < -2 \end{array}\right\} \Rightarrow$$

$$x = \left\{\begin{array}{c} \pm\frac{a^{\frac{a}{2}+1}}{(\frac{a}{2}+1)} \\ \left(\frac{1 \pm \sqrt{5}}{2}\right) \log x \\ \frac{x^{a+2}}{(a+1)(a+2)} \end{array}\right\} \Rightarrow y = \left\{\begin{array}{c} e^{\pm\frac{a^{\frac{a}{2}+1}}{(\frac{a}{2}+1)}} \\ x^{\frac{1 \pm \sqrt{5}}{2}} \\ e^{\frac{x^{a+2}}{(a+1)(a+2)}} \end{array}\right\} \quad x \to +\infty$$

Note that when $a < -2$

$$y \sim e^{\frac{x^{a+2}}{(a+2)(a+1)}} \text{ but a} + 2 < 0$$

so $y \to const$ as $x \to +\infty$
When $a = -2,\ y \sim x^{\frac{1 \pm \sqrt{5}}{2}}$, algebraic growth (modified at higher orders by an exp. which $\to const$ as $x \to +\infty$)
Question: Where's the second solution for (iii)?
Hint: Think higher order $\cdots$