Question

Consider the equation

$$xy'' - (x+2)y = 0.$$

(a) Find the first few terms of the solutions as $x \to +\infty$ by the method of dominant balance.

(b) Show that one solution is exactly $x\exp(x)$, and demonstrate that this is consistent with your answer to part (a).

Answer

$y'' - \left(1 + \frac{2}{x}\right) y = 0$

(a) Try ansatz $y \sim e^{(\phi_0(x)+\phi_1(x)+\phi_2(x)+\cdots)}\{\phi_r(x)\}$

forming an asymptotic sequence as $x \to +\infty$

$y' \sim \{\phi_0' + \phi_1' + \phi_2' + \cdots\}e^{\{\phi_0+\phi_1+\phi_2+\cdots\}} \quad x \to +\infty$

$y'' \sim \{(\phi_0'' + \phi_1'' + \phi_2'' + \cdots) + (phi_0' + \phi_1' + \phi_2' + \cdots)^2\}e^{\{\phi_0+\phi_1+\phi_2+\cdots\}}$

$x \to +\infty$

Substitute into equation and simplify

$(\phi_0'' + \phi_1'' + \phi_2'' + \cdots) + (phi_0' + \phi_1' + \phi_2' + \cdots)^2 = 1 + \frac{2}{x}$

Assume $\{\phi_r''(x)\}$ and $\{\phi_r'(x)\}$ form an asymptotic sequences as $x \to +\infty$ as well.

<u>First balance</u>

$$\phi_0'' + {\phi_0'}^2 = 1$$

$\left\{ \begin{array}{l} \phi_0\phi_r' = o({\phi_0'}^2) \\ \phi_r'' = o(\phi_0'') \end{array} \right.$ by asymptotic sequences.

The only balance which works is

$$\begin{array}{rl} {\phi_0'}^2 = 1 & \Rightarrow \quad \phi_0'' = 0 = o(1) \quad x \to +\infty \\ \text{Therefore } \phi_0' = \pm 1 & \Rightarrow \quad \phi_0 = \pm x + \underbrace{const}_{\text{absorb into arbitrary prefactor of exponential.}} \surd\surd \end{array}$$

Second balance

ϕ_1'

$$
\begin{aligned}
(0 + \phi_1'' + \cdots) + (\overbrace{\pm 1} + \phi_1' + \cdots)^2 \quad &= \quad 1 + \frac{2}{x} \\
\Rightarrow \phi_1'' + 1 \underbrace{\pm 2\phi_1'} + \phi_1^2 + \cdots \& = \quad 1 + \frac{2}{x}
\end{aligned}
$$

$|2\phi_0\phi_1| \gg \phi_1^2$ by asymp. sequence assump.

$$\text{Therefore } \phi_1'' \pm 2\phi_1' = \frac{2}{\mathrm{x}}$$

The only balance which works is

$\pm 2\phi_1' = \frac{2}{x} \Rightarrow \phi_1 = \pm \log x$

$\left(\phi_1'' = \mp \frac{1}{x^2} = o\left(\frac{1}{x}\right),\ x \to +\infty\right)$

Third balance:

$\left(\underbrace{0}_{\phi_0''} \mp \frac{1}{x^2} + \phi_2'' + \cdots\right) + (\phi_0' + \phi_1' + \phi_2' + \cdots)^2 = 1 + 2??$

$\left(\mp \underbrace{\frac{1}{x^2}}_{\phi_1''} + \phi_2''\right) + \left(\underbrace{1}_{{\phi_0'}^2} + \underbrace{\frac{2}{x}}_{2\phi_0'\phi_1'} + {\phi_1'}^2 + 2\phi_0'\phi_2' + \cdots\right) = 1 + \frac{2}{x}$

By asymptotic sequence assumptions $\phi_2'' = O(\phi_1'') = \mp \frac{1}{x^2}$

Thus we have

$$\mp \frac{1}{x^2} + ({\phi_1'}^2 + \underbrace{2\phi_0'\phi_2'} + \cdots) = 0$$

$$\phi_1'\phi_2' = o(\text{all of this}) \text{ so neglect it}$$

Now we can't make much of a statement about whether ${\phi_1'}^2$ dominates $\phi_0'\phi_2'$. Why? $|\phi_0| > |\phi_1| > |\phi_2|$, but $|phi_1^2|$ could be $> |\phi_0'\phi_2'|$ if $\phi_0',\ \phi_2'$ is small enough, or $|{\phi_1'}^2|$ could be $< |\phi_0'\phi_2'|$ if $\phi_0',\ \phi_2'$ are large enough.

Thus we should keep both.

${\phi_1'}^2 = \frac{1}{x^2},\ \phi_0' = \pm 1$

Therefore $\mp \frac{1}{x^2} + \frac{1}{x^2} \pm 2,\ \phi_2' = 0$

$\Rightarrow \phi_2' = 0$ or $+\frac{1}{x^2}$

$\Rightarrow \phi_2' = const$ or $-\frac{1}{x}$

The constant value can be absorbed into the exp. prefactor so set $const = 0$ without loss of generality.

Therefore pulling everything together, we have

$y \sim A \exp(+x + \log x + 0 + o(\log x)) + B \exp\left(-x - \log x - \frac{1}{x} + o\left(\frac{1}{x}\right)\right)\ x \to +\infty$

$$\Rightarrow y \sim Axe^{x+o(\log x)} + \frac{B}{x}e^{-x-\frac{1}{x}+o(\frac{1}{x})},\ A,\ B consts,\ x \to +\infty$$

(b) Set $y = xe^x,\ y' = e^x(1+x),\ y'' = e^x(2+x)$

Therefore $xy'' = xe^x(2+x) = (x+2)y$

$\Rightarrow xy'' - (x+2)y = 0\surd\surd$

So $y = xe^x$ is a solution. This is consistent with (a) as the $\phi_2 = 0$ solution gives this behaviour as $x \to +\infty$. By careful consideration of boundary data it can be established that $\phi_r = 0$ for all $r \geq 2$ is an asymptotic solution $\Rightarrow$ an exact one as well.

Moral: asymptotic methods may give the exact result if they truncate at some point.

($\Rightarrow$ their divergence is intimately linked to the $\sum_{r=0}^{\infty}$ (∞ series) representation)