Application to Vibrating String

We have a uniform string of length l, mass per unit length m, and equilibrium tension T. Let there be a disturbing (transverse) force $f(x,t)$ per unit length. With the assumptions that

1) There are no longitudinal body forces (or negligible forces).
2) Displacement $y(x,t)$ is purely transverse.
3) $\frac{\partial y}{\partial x}$ is small compared with 1.

it can be shown that to the first order T is uniform along the string, and the equation of motion is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^y}{\partial t^2} - \frac{1}{T}f(x,t) \tag{1}$$

when $c = \left(\frac{T}{m}\right)^{\frac{1}{2}}$

For free motion $f = 0$ and we have

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \tag{2}$$

Seeking solutions of the form

$$y = X(x)Y(t)$$

$$\frac{1}{X}\frac{\partial^X}{dx^2} = \frac{1}{c^2}\frac{1}{Y}\frac{d^2Y}{dt^2}$$

Hence both sides are constant and we write the constant as $-\frac{w^2}{c^2}$.

This gives $\frac{d^2Y}{dt^2} = -w^2 Y$

$$Y = A\cos wt + b\sin wt \tag{3}$$

and $\frac{d^2X}{dt^2} = -\frac{w^2}{c^2}X$ (4)

$$X = C\cos\frac{w}{c}x + D\sin\frac{w}{c}x \tag{4a}$$

For fixed ends we require $y(0,t) = y(l,t) = 0$

$$\text{Therefore} \quad \left.\begin{array}{l} X(0) = 0 \\ X(l) = 0 \end{array}\right\} \tag{5}$$

The differential equation (4), and the conditions (5) constitute a two-point boundary problem. The differential equation includes a parameter $\lambda = \dfrac{w^2}{c^2}$ not yet determined.

In fact we show below that λ must have one of a set of values

$\lambda_1 = \dfrac{\pi^2}{l^2},\ \lambda_2 = \dfrac{(2\pi)^2}{l^2} \cdots$.

These are the eigenvalues, the corresponding solutions $X(x, \lambda)$ are the eigenfunctions.

i.e. (4) and (5) constitute an eigenvalue problem.

$X = C \cos \dfrac{w}{c} x + D \sin \dfrac{w}{c} x$

The conditions (5) give $C = 0$ and $D \sin \dfrac{wl}{c} = 0$

Hence for a non-trivial solution $\sin \dfrac{wl}{c} = 0$, therefore $w = \dfrac{n\pi c}{l}$ $\quad n = 1, 2 \cdots$

Hence solutions satisfying (5) are

$$y = \sin \frac{n\pi x}{l} \left[A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right] \tag{6}$$

These are the normal modes of vibration and the values of $\dfrac{w}{2\pi}$, i.e. $\dfrac{c}{2l}, \dfrac{2c}{2l} \cdots$ are the normal frequency.

Formally a general solution satisfying (2) and the end conditions (5) is

$$y = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \left[A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right]$$

If this series converges and is twice differentiable term-by-term then the function y is a solution.

Given initial conditions

Suppose that $\dot{y}(x,0) = 0$ and $y(x,0) = F(x) \qquad 0 \leq x \leq l$

Now a solution satisfying $\dot{y}(x,0) = 0$ is

$$y = \sum_{1}^{\infty} A_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \tag{1}$$

when $t = 0$ this gives $y(x,0) = \displaystyle\sum_{1}^{\infty} A_n \sin \frac{n\pi x}{l}$.

Hence $\sum_1^\infty A_n \sin \frac{n\pi x}{l}$ is the sine series expansion of $F(x)$ in $0 \leq x \leq l$.

i.e. $A_n = \frac{2}{l}\int_0^l F(x) \sin \frac{n\pi x}{l} dx$ (2)

Interpretation of solutions in terms of progressive waves

We have $y = \frac{1}{2}\sum_1^\infty A_n \left[\sin \frac{n\pi}{l}(x+ct) + \sin \frac{n\pi}{l}(x-ct)\right]$

Write $F_s(x) = \sum_1^\infty A_n \sin \frac{n\pi x}{l}$

$$F_s(x) = \begin{cases} F(x) & 0 \leq x \leq l \\ -F(-x) & -l \leq x \leq 0 \end{cases}$$

$F_s(x+2l) = F_s(x)$

$y = \frac{1}{2}\{F_s(x+ct) + F_s(x-ct)\}$

$\frac{\partial^2 y}{\partial t^2} = \frac{1}{2}\{F_s''(x+ct) + F_s''(x-ct)\}$

$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{1}{2}\{F_s''(x+ct) + F_s''(x-ct)\}$

Whenever $\zeta = x+ct$ and $\eta = x-ct$ are such that $F(x)$ is twice differentiable at the values concerned.

Also $y(x,0) = \frac{1}{2}\{F_s(x) + F_s(x)\} = F(x)$

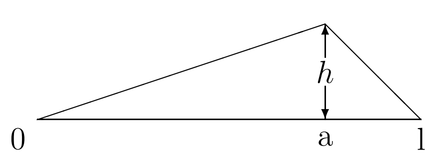
$y(0,t) = 0 \qquad (F_s(x) \text{ odd})$

$y(l,y) = \frac{1}{2}\{F_s(l+ct) + F_s(l-ct)\}$

$\qquad = \frac{1}{2}\{F_s(l+ct) + F_s(-[l+ct])\} = 0 \qquad (F \text{ periodic } 2l).$

Plucked String

$$F(x) = \begin{cases} \frac{h}{a}x & 0 \leq x \leq a \\ \frac{h}{l-a}(l-x) & a \leq x \leq l \end{cases}$$



$$
\begin{aligned}
A_n &= \frac{2}{l}\int_0^l F(x)\sin\frac{n\pi x}{l}dx \\
&= \frac{2}{l}\left[-\frac{F(x)}{\frac{n\pi}{l}}\cos\frac{n\pi x}{l}\right]_0^l + \frac{2}{l}\int_0^l F'(x)\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}dx \\
&= 0 + \frac{2}{l}\left[F'(x)\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]_0^a + \frac{2}{l}\left[F'(x)\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]_a^l - \int_0^l F''(x)\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}dx \\
&= \frac{2}{l}\frac{l^2}{n^2\pi^2}\sin\frac{n\pi a}{l}[f'(a-0) - f'(a+0)] \qquad \text{as } f'' \equiv 0 \\
&= \frac{2l}{n^2\pi^2}\sin\frac{n\pi a}{l}\left[\frac{h}{a} + \frac{h}{l-a}\right] \\
&= \frac{2hl^2}{n^2\pi^2 a(l-a)}\sin\frac{n\pi a}{l}
\end{aligned}
$$

Therefore $y(x,t) = \dfrac{2l^2h}{\pi^2a^2(l-a)}\displaystyle\sum_1^\infty \frac{\sin\frac{n\pi x}{l}\sin\frac{n\pi a}{l}}{n^2}\cos\frac{n\pi ct}{l}$

Note that any normal mode which has a node at $x = a$ is absent from the series since in that case $\sin\dfrac{n\pi a}{l} = 0$ for all n.

This occurs when $\dfrac{a}{l} =$ rational $\dfrac{r}{s}$ $\quad r < s$.

Then $\sin\dfrac{n\pi a}{l}$ vanishes for $n = s, 2s, 3s\cdots$.

The corresponding modes are absent.

Physical Illustration of Parseval's Theorem

$$K.E = \frac{1}{2}\int_0^l m\left(\frac{\partial y}{\partial t}\right)^2 dx$$

The P.E of the element $\triangle x$ is the work done by the tension at the ends in extending the element from $\triangle x$ to $\left(\triangle x^2 + \triangle y^2\right)^{\frac{1}{2}}$

i.e. to $\triangle x\left(1 + \left(\frac{\triangle y}{\triangle x}\right)^2\right)^{\frac{1}{2}} = \triangle x\left(1 + \frac{1}{2}\left(\frac{\triangle y}{\triangle x}\right)^2 + \cdots\right)$

Therefore the extension is $\dfrac{1}{2}\left(\dfrac{\partial y}{\partial x}\right)^2 dx$

The work done is $\frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 dx$

Therefore the total work done is $\frac{1}{2}\int_0^l T\left(\frac{\partial y}{\partial x}\right)^2 dx$

$y = \sum_1^\infty C_n \sin\frac{n\pi x}{l}\cos\left(\frac{n\pi ct}{l}+\alpha_n\right)$

$\frac{\partial y}{\partial t} = \sum_1^\infty -\frac{n\pi c}{l}\sin\frac{n\pi x}{l}C_n\sin\left(\frac{n\pi ct}{l}+\alpha_n\right)$

$\frac{\partial y}{\partial x} = \sum_1^\infty \frac{n\pi}{l}\cos\frac{n\pi x}{l}C_n\cos\left(\frac{n\pi ct}{l}+\alpha_n\right)$

Applying Parseval's formula

$$\begin{aligned}
\int_0^l \left(\frac{\partial y}{\partial t}\right)^2 dx &= \frac{\pi^2 c^2}{l^2}\sum_1^\infty n^2\left(C_n\sin\left[\frac{n\pi ct}{l}+\alpha_n\right]\right)^2\frac{l}{2} \\
\int_0^l \left(\frac{\partial y}{\partial t}\right)^2 dx &= \frac{\pi^2}{l^2}\sum_1^\infty n^2\left(C_n\cos\left[\frac{n\pi ct}{l}+\alpha_n\right]\right)^2\frac{l}{2}
\end{aligned}$$

Hence $\quad K.E = \frac{1}{2}m\frac{\pi^2c^2}{l^2}\sum_1^\infty n^2C_n^2\sin^2\left(\frac{n\pi ct}{l}+\alpha_n\right)\frac{l}{2}$

$T = mc^2$ so $P.E = \frac{1}{2}m\frac{\pi^2c^2}{l^2}\sum_1^\infty n^2C_n^2\cos^2\left(\frac{n\pi ct}{l}+\alpha_n\right)\frac{l}{2}$

Sum$=\frac{1}{4}\frac{m\pi^2c^2}{l}\sum_1^\infty n^2C_n^2=$ constant$=\bar{K.E}+\bar{P.E}$

Forced Motion

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial y}{\partial t^2} - f(x,t)T \qquad (1)$$

$f(x,t)=$ force / unit length at distance x, and time t.

Assume a simple harmonic forcing term $f(x,t) = F(x)\cos wt$.

We seek a solution of (1), simple harmonic with the same frequency.

i.e. $y = Y(x)\cos wt \qquad (2)$

By substitution we find that with $G(x) = -\frac{F(x)}{T}$

$$\left.\begin{array}{l}\frac{d^2Y}{dx^2}+\frac{w^2}{c^2}Y = G(x)\\ \text{We must also have } Y(0)=0=Y(l)\end{array}\right\} \qquad (3)$$

We seek a solution in which Y and Y' are continuous, and we shall assume that $G(x)$ is continuous in $0 \leq x \leq l$.
[Note that the general solution of (1) is of the form $Y(x)\cos wt + z$, where z satisfies the homogeneous equation $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^z}{\partial t^2}$ this added term would be needed in general, in order that the initial conditions should be satisfied, since the particular solution $Y(x)\cos wt$, would not, in general, satisfy these.]

Write $\lambda = \dfrac{w^2}{c^2}$

$\lambda_n = \left(\dfrac{n\pi}{l}\right)^2$ $\qquad \left[w_n = \dfrac{n\pi c}{l},\ \text{i.e } \lambda_n = \dfrac{wn^2}{c^2}\right]$

Then (3) is

$$\left.\begin{aligned} Y'' + \lambda Y &= G(x) \\ Y(0) &= 0 \\ Y(l) &= 0 \end{aligned}\right\} \tag{3a}$$

Write $u_n = \sin \dfrac{n\pi x}{l}$, so that

$$u_n'' + \lambda_n u_n = 0 \tag{4}$$

From $(3a)u_n - (4)Y$ we get

$u_n Y'' - Y u_n'' + (\lambda - \lambda_n) Y u_n = G u_n$

i.e. $\dfrac{d}{dx}(u_n Y' - u_n' Y) + (\lambda - \lambda_n) u_n Y = G u_n$

integrating from 0 to l, we have that

$[u_n Y' - u_n' Y]_0^l + (\lambda - \lambda_n)\displaystyle\int_0^l u_n Y dx = \int_0^l u_n G dx$

[The evaluation of the first term is between the end limits only as Y, Y', u_n, u_n' are continuous in $[0, l]$.]

$u_n(0) = Y(0) = u_n(l) = Y(l) = 0$ therefore $[u_n Y' - u_n' Y]_0^l = 0$

Therefore $(\lambda - \lambda_n)\displaystyle\int_0^l u_n Y dx = \int_0^l u_n G dx$ (5)

Case I $\lambda \neq \lambda_n \qquad (n = 1, 2, \cdots)$

i.e. λ_n is not an eigenvalue of the system

$Y'' + \lambda Y = 0 \qquad Y(0) = 0 \qquad Y(l) = 0$

$$\int_0^l u_n Y dx = \frac{1}{\lambda - \lambda_n}\int_0^l u_n G dx \tag{6}$$

i.e. $Y_n = \dfrac{1}{\lambda - \lambda_n} G_n$ where Y_n and G_n are the Fourier sine coefficients for Y and G in $[0, l]$.

Hence if $G(x) = \sum_1^\infty G_n \sin \frac{n\pi x}{l}$

then $Y(x) = \sum_1^\infty \frac{G_n}{\lambda - \lambda_n} \sin \frac{n\pi x}{l} = \sum_1^\infty \frac{c^2 G_n}{w^2 - w_n^2} \sin \frac{n\pi x}{l}$ is (formally) a solution of the differential equation and the end conditions.

[Note that if $G(x)$ is continuous in $0 \leq x \leq l$ and $G(0) = 0, \; G(l) = 0$, then the coefficients are at most $O(\frac{1}{n^2})$ and the coefficients of $Y(x)$ are at most $O(\frac{1}{n^4})$. Hence the series for $Y(x)$ is certainly twice differentiable term-by-term since the derived series has coefficients of order $\frac{1}{n^2}$ and hence converges absolutely and uniformly.]

Note that when w is near to w_m the dominant term in Y is then

$\frac{c^2}{w^2 - w_m^2} G_m \sin \frac{n\pi x}{l}$ if $G_m \neq 0$.

When $w = w_m$ the solution fails unless $G_m = 0$.

Case II $\lambda = \lambda_m$ i.e. $w = w_m$

From $\lambda - \lambda_n \int_0^l Y_{u_n} ax = \int_0^l G u_n dx$

We have $0 = \int_0^l G u_m dx = \frac{l}{2} G_m$

Therefore $G_m = 0$ is a necessary condition for the existence of a solution of the type $y = Y(x) \cos wt$.

If $G_m = 0 \qquad Y_n = \frac{1}{\lambda_m - \lambda_n} G_n \qquad n \neq m$

Consider $Y(x) = \sum_{n=1,\, n\neq m}^\infty \frac{1}{\lambda_m - \lambda_n} G_n \sin \frac{n\pi x}{l}$

By formal differentiation term by term

$$\left(\frac{d^2}{dx^2} + \lambda_m\right) Y = \sum_{n=1,\, n\neq m}^\infty G_n \sin \frac{n\pi x}{l} = G(x) \qquad \left(\frac{n^2\pi^2}{l^2} = \lambda_n\right)$$

Since the series is the sine series for $G(x)$ where there is no term in $\sin \frac{m\pi x}{l}$.

Hence the above expression for $Y(x)$ is a solution, but is not unique since

$Y(x) = \sum_{n=1,\, n\neq m}^\infty \frac{G_n}{\lambda_m - \lambda_n} \sin \frac{n\pi x}{l} + A \sin \frac{m\pi x}{l}$

is also a solution.

[In case I the solution for Y is unique, for if Y_1 and Y_2 satisfy

$Y'' + \lambda Y = G, \;\; Y(0) = Y(l) = 0$, then putting $Y_3 = Y_1 - Y_2$,

$Y_3'' + \lambda Y = 0$, $Y_3(0) = Y_3(l) = 0$. This has a non trivial solution only if $\lambda = \lambda_n$ for some n, which is not so, i.e $Y_3 = 0$.]

Summary

i) $\lambda \neq \lambda_n$ for every n, the solution for Y exists and is unique.

ii) $\lambda = \lambda_m$, no solution $y = Y \cos wt$ when $G_m \neq 0$. If $G_m = 0$ a solution exists but is not unique.

Case III $\lambda = \lambda_m \qquad G_m \neq 0$

The solution $Y(x) = \sum_1^{\infty}{}' \dfrac{G_n}{\lambda_n - \lambda_m} \sin \dfrac{n\pi x}{l}$ fails.

If $\lambda \neq \lambda_m$ for the moment,

$$y(x,t) = \left[\sum_{n=1,\, n\neq m}^{\infty} \frac{G_n}{\lambda - \lambda_m} \sin \frac{n\pi x}{l} \right] \cos wt + G_m \sin \frac{m\pi x}{l} \frac{\cos wt}{\lambda - \lambda_m}$$

Consider $y_1(x,t) = y(x,t) - G_m \sin \dfrac{m\pi x}{l} \dfrac{\cos w_m t}{\lambda - \lambda_m}$

where the added term satisfies

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) y = 0 \qquad y(0,t) = y(l,t) = 0$$

and hence does not alter the forcing term $G(x) \cos wt$. Then

$$y_1(x,t) = \left[\sum_{n=1,\, n\neq m}^{\infty} \frac{G_n}{\lambda - \lambda_m} \sin \frac{n\pi x}{l} \right] \cos wt + G_m \sin \frac{m\pi x}{l} \frac{\cos wt - \cos w_m t}{\lambda - \lambda_m}$$

Now $\displaystyle\lim_{\lambda \to \lambda_m} \frac{\cos wt - \cos w_m t}{\lambda - \lambda_m} = \frac{\partial}{\partial \lambda} \frac{(\cos wt - \cos w_m t)_{\lambda = \lambda_m}}{l}$

$$= -t \sin w_m t \left(\frac{dw}{d\lambda} \right)_{\lambda = \lambda_m}$$

but $\lambda = \dfrac{w^2}{c^2}$ therefore $\dfrac{2}{w_m} \left(\dfrac{dw}{d\lambda} \right)_{\lambda = \lambda_m} = \dfrac{1}{\lambda_m}$

Therefore the above limit is $-t \sin w_m t \dfrac{w_m}{2\lambda_m} = -\dfrac{(w_m t) \sin(w_m t)}{2\lambda_m}$

Hence the limiting form of $y_1(x,t)$ is

$$\left[\sum_{n=1,\, n\neq m}^{\infty} \frac{G_n}{\lambda - \lambda_m} \sin \frac{n\pi x}{l} \right] \cos w_m t - \frac{G_m}{2} \sin \frac{m\pi x}{l} \frac{w_m t \sin w_m t}{\lambda_m}$$

This shows the phenomenon of resonance since the second term has amplitude increasing with time (linearly).

Alternative method for solution of the non-homogeneous equation (in case $\lambda \neq \lambda_n$)

$Y'' + \lambda Y = G(x) \qquad 0 \leq x \leq l \qquad Y(0) = y(l) = 0 \qquad (1)$

Let u, v be solutions of the homogeneous equation

$u'' + \lambda u = 0 \qquad (2a)$

$v'' + \lambda v = 0 \qquad (2b)$

Choose u and v so that

$u(0) = v(l) = 0 \qquad (3)$

In this case $u = \sin \lambda^{\frac{1}{2}} x \quad (\sin \frac{wx}{l}) \quad$ and $\quad v = \sin \lambda^{\frac{1}{2}}(l - x) \quad (\sin \frac{w}{c}(l - x))$

From $(1)u - (2a)Y$ we get $uY'' - u''Y = uG$

i.e. $\dfrac{d}{dx}(uY' - u'Y) = u(x)G(x) \qquad (4)$

Similarly $(1)v - (2b)Y$ gives $\dfrac{d}{dx}(vY' - v'Y) = v(x)G(x) \qquad (5)$

Finally $(2a)v - (2b)u$ gives $\dfrac{d}{dx}(vu' - uv') = 0 \qquad (6)$

From (6) $v(x)u'(x) - v'(x)u(x) = \text{const} = v(0)u'(0)$

$= \lambda^{\frac{1}{2}} \sin \lambda^{\frac{1}{2}} l = \dfrac{w}{c} \sin \dfrac{wl}{c} = \triangle(\lambda) \qquad (7)$

Integrate (4) from 0 to x:

$u(x)Y'(x) - u'(x)Y(x) = \displaystyle\int_0^x u(\zeta)G(\zeta)d\zeta \qquad (8)$

since $u(0) = Y(0) = 0$

Integrate (5) from l to x:

$v(x)Y'(x) - v'(x)Y(x) = \displaystyle\int_l^x v(\zeta)G(\zeta)d\zeta \qquad (9)$

since $v(l) = Y(l) = 0$.

Equations (8) and (9) are linear equations in Y and Y'.

The determinant of the coefficients is

$\begin{vmatrix} u(x) & -u'(x) \\ v(x) & -v'(x) \end{vmatrix} = u'(x)v(x) - v'(x)u(x) = \triangle(\lambda) \qquad$ from (7)

Hence (8) and (9) can be solved for Y and Y' (for any $G(x)$) if

$\triangle(\lambda) \neq 0, \quad \triangle = \lambda^{\frac{1}{2}} \sin \lambda^{\frac{1}{2}} l \neq 0$ as λ is not an eigenvalue.

Hence solving for $Y(x)$

$(9)u - (8)v$ gives

$\triangle(\lambda)Y(x) = u(x) \displaystyle\int_l^x v(\zeta)G(\zeta)d\zeta - v(x) \int_0^x u(\zeta)G(\zeta)d\zeta \qquad (10)$

Similarly we have

$$\triangle(\lambda)Y'(x) = u'(x)\int_l^x v(\zeta)G(\zeta)d\zeta - v'(x)\int_0^x u(\zeta)G(\zeta)d\zeta \qquad (10')$$

[Note that (10′) follows from differentiation of 10]

Differentiating (10′) we find

$$\triangle(\lambda)Y''(x) = u''(x)\int_l^x v(\zeta)G(\zeta)d\zeta - v''(x)\int_0^x u(\zeta)G(\zeta)d\zeta$$
$$+[u'(x)v(x) - v'(x)u(x)]G(x) \qquad (10'')$$

Therefore $\triangle\lambda(Y'' + \lambda Y) = (u'' + \lambda u)\int_l^x -(v'' + \lambda v)\int_0^x + \triangle(\lambda)G(x)$
$= \triangle(\lambda)G(x)$ as $u'' + \lambda u = v'' + \lambda v = 0$

Since $\triangle(\lambda) \neq 0 \qquad Y'' + \lambda Y = 0.$

We can write 10 as

$$\triangle(\lambda)Y(x) = -\int_0^l g(x,\zeta)G(\zeta)d\zeta \qquad (10a)$$

$$g(x,\zeta) = \begin{cases} \frac{1}{\triangle(\lambda)}v(x)u(\zeta) & 0 \leq \zeta \leq x \\ \frac{1}{\triangle(\lambda)}v(\zeta)u(x) & x \leq \zeta \leq l \end{cases}$$

$g(x,\zeta)$ is continuous in ζ at x.

$\frac{\partial}{\partial\zeta}g(x,\zeta)$ is discontinuous at x.

$$\left[\frac{\partial}{\partial\zeta}(g(x,\zeta))\right]_{x-0}^{x+0} = \frac{1}{\triangle(\lambda)}[v'(\zeta)u(x) - v(x)u'(\zeta)]_{x=\zeta} = -1$$

$\left(\frac{\partial 2}{\partial\zeta^2} + \lambda\right) g = 0$ also $g(\zeta,x) = g(x,\zeta)$ since

$$g(x,\zeta) = \frac{1}{\triangle\lambda}[v(\max(\zeta,x)) \cup (\min(x,\zeta))]$$

for $0 \leq x \leq l, \quad 0 \leq \zeta \leq l$, and $\max(x,\zeta) = \max(\zeta,x)$.