Solutions of Laplace's Equation and others in Spherical Co-ordinates

$\bigtriangledown^2 V = 0$ Laplace's Equation

$\bigtriangledown^2 V = \dfrac{1}{c^2}\dfrac{\partial^2 V}{\partial t^2}$ Wave Equation

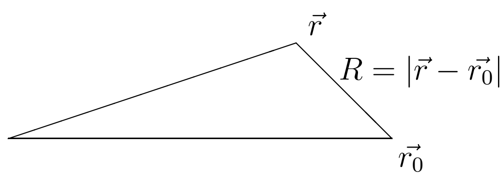
$\bigtriangledown^2 V = \dfrac{1}{K}\dfrac{\partial V}{\partial t}$ Diffusion Equation

$\bigtriangledown^2 \psi + \{l - v(x,y,z)\}\psi = 0$ Wave Mechanics Equation

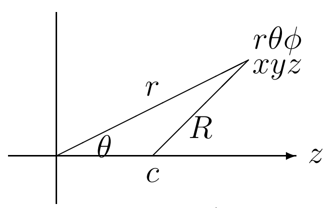
$$\bigtriangledown^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Axially Symmetric Solutions of $\bigtriangledown^2 V = 0$

$\dfrac{1}{R} = \dfrac{1}{|\vec{r} - \vec{r_0}|}$ is a solution of Laplaces equation in the coordinates x, y, z.



Consider $\dfrac{1}{r} = \dfrac{1}{(r^2 + c^2 - 2cr\cos\theta)^{\frac{1}{2}}} = \dfrac{1}{(x^2 + y^2 + (z-c)^2)^{\frac{1}{2}}}$



We have $\bigtriangledown^2\left(\dfrac{1}{R}\right) = 0$ and so $\dfrac{\partial^n}{\partial c^n}\bigtriangledown^2\dfrac{1}{r} = 0$ or $\bigtriangledown^2\dfrac{\partial^n}{\partial c^n}\dfrac{1}{R} = 0$

i.e. $\dfrac{1}{n!}\dfrac{\partial^n}{\partial c^n}\left(\dfrac{1}{R}\right)$ is a solution of Laplaces equation and in particular

$\dfrac{1}{n!}\left[\dfrac{\partial^n}{\partial c^n}\left(\dfrac{1}{R}\right)\right]_{c=0}$ is a solution.

[N.B. $\frac{\partial}{\partial c}(\frac{1}{R}) = -\frac{\partial}{\partial z}(\frac{1}{R})$ and so $[\frac{\partial^n}{\partial c^n}(\frac{1}{r})]_{c=0} = (-1)^n\frac{\partial^n}{\partial z^n}\frac{1}{r}$ i.e. the above solution can be written $\frac{(-1)^n}{n!}\frac{\partial^n}{\partial c^n}\frac{1}{r}$]

$\dfrac{1}{R} = \dfrac{1}{[(r-ce^{i\theta})(r-ce^{-i\theta})]^{\frac{1}{2}}}$ and $(r-ce^{i\theta})^{-\frac{1}{2}}$ has a power series expansion in powers of c, which is absolutely convergent for $\left|\dfrac{ce^{i\theta}}{r}\right| < 1$ i.e. for $\dfrac{|c|}{r} < 1$ when θ is real.
Similarly for $(r-ce^{-i\theta})^{-\frac{1}{2}}$.
Therefore $\dfrac{1}{R} = \dfrac{1}{(r-ce^{i\theta})^{\frac{1}{2}}(r-ce^{-i\theta})^{\frac{1}{2}}}$ has a power series expansion in c which is also convergent for $\dfrac{|c|}{r} < 1$ (θ real) and the coefficient of c^n is $\dfrac{1}{n!}\left[\dfrac{\partial^n}{\partial c^n}\dfrac{1}{R}\right]_{c=0}$.
Therefore the coefficient of c^n in the above expansion of $\frac{1}{R}$ in powers of c is a solution of Laplace's equation
$\dfrac{1}{R} = \dfrac{1}{r\left[1-\frac{2c}{r}\cos\theta+\frac{c^2}{r^2}\right]^{\frac{1}{2}}}$ and so $\dfrac{1}{R} = \dfrac{1}{r}\sum_{n=0}^{\infty}\left(\dfrac{c}{r}\right)^n P_n(\cos\theta)$ for $\dfrac{|c|}{r} < 1$ and θ real.
Thus $\dfrac{P_n(\cos\theta)}{r^{n+1}}$ is a solution of Laplace's equation and is axially symmetric.
Similarly, by considering the expansion of $\frac{1}{R}$ for positive powers of $\frac{1}{c}$ with $|c| > r$, θ real, we find that $r^n P_n(\cos\theta)$ is also an axially symmetric solution.

Alternative Argument

If V is a solution of $\bigtriangledown^2 V = 0$, homogeneous and of degree n, then $\dfrac{V}{r^{2n+1}}$ is also a solution of $\bigtriangledown^2 V = 0$ of degree $-(n+1)$.

Proof

$\bigtriangledown^2 V r^m = r^m \bigtriangledown^2 V + V \bigtriangledown^2 r^m + 2\vec{\bigtriangledown} V \vec{\bigtriangledown} r^m = r^m \bigtriangledown^2 V + Vm(m+1)r^{m-2} + 2mr^{m-1}\frac{\partial V}{\partial r}$

and since V is homogeneous of degree n, $r\dfrac{\partial V}{\partial r} = nV$. So if $\bigtriangledown^2 V = 0$, then $\bigtriangledown^2 V r^m = r^{m-2}[m(m+1)+2mn]V = 0$ if $m = 0, -2n-1$.

From $\dfrac{1}{(r^2+c^2-2rc\cos\theta)^{\frac{1}{2}}} = \dfrac{1}{r}\sum_{n=0}^{\infty}\left(\dfrac{c}{r}\right)^n P_n(\cos\theta)$ $\quad (|c| < r,\ \theta \text{ real})$

Putting $r = 1,\ c = h,\ \cos\theta = \mu$, we have the definition of the P_n's.

$$\frac{1}{(1-2\mu h+h^2)^{\frac{1}{2}}} = \sum_{n=0}^{\infty} h^n P_n(\mu) \qquad (|h| < 1, \mu \text{ real } -1 \le \mu \le 1)$$

This expansion is valid for all h and μ, where $|h| < |\mu \pm (\mu^2-1)^{\frac{1}{2}}|$, since

$(\mu + (\mu^2 - 1)^{\frac{1}{2}} - h)(\mu - (\mu^2 - 1)^{\frac{1}{2}} - h) = 1 - 2\mu h + h^2$

Properties of $\mathbf{P_n}(\mu)$

1) $P_n(\mu)$ is a polynomial in μ of degree n, in alternate powers $n, n-2, \cdots 1$ or 0. i.e.$P_{2n}(\mu)$ contains even powers and is an even function. $P_{2n+1}(\mu)$ contains odd powers and is an odd function.

2) $P_n(1) = 1 \qquad P_n(-1) = (-1)^n$

3) $|P_n(\mu)| \le 1 \qquad -1 \le \mu \le 1$

4) Legendre's Equation

   $P_n(\mu)$ is a solution of $\dfrac{d}{d\mu}(1-\mu^2)\dfrac{d}{d\mu}w + n(n+1)w = 0$

5) Orthogonal Properties

$$\int_{-1}^{1} P_n(\mu)P_m(\mu)d\mu = \begin{cases} 0 & m \neq n \\ \frac{2}{2n+1} & m = n \end{cases}$$

6) Rodriguez's Formula

$$P_n(\mu) = \frac{1}{2^n n!}\frac{d^n}{d\mu^n}(\mu^2 - 1)^n$$

7) Recurrence Formulae

$$(n+1)P_{n+1}(\mu) - (2n+1)\mu P_n(\mu) + nP_{n-1}(\mu) = 0$$
$$P'_{n+1}(\mu) - P'_{n-1}(\mu) = (2n+1)P_n(\mu)$$
$$(\mu^2 - 1)P'_n(\mu) = n[\mu P_n(\mu) - P_{n-1}(\mu)]$$

Proofs

1) Write $c_0 = 1$, $c_r = \dfrac{1.3\cdots 2r-1}{2.4\cdots 2r} = \dfrac{\Gamma\left(r+\frac{1}{2}\right)}{r!\Gamma\left(\frac{1}{2}\right)}$

   Then $(1-z)^{-\frac{1}{2}} = \sum_0^\infty c_r z^r \qquad |z| < 1$

$$\frac{1}{(1+h^2-2\mu h)^{\frac{1}{2}}} = \frac{1}{[(1-he^{i\theta})(1-he^{-i\theta})]^{\frac{1}{2}}}$$

where $\cos\theta = \mu$ (θ real if $-1 \le \mu \le 1$, but we don't assume this).

So $\dfrac{1}{(1+h^2-2\mu h)^{\frac{1}{2}}} = \sum_0^\infty c_r h^r e^{ir\theta} \sum_0^\infty c_s h^s e^{is\theta} \qquad |h| < |\mu \pm (\mu^2-1)^{\frac{1}{2}}|$

$P_n(\mu)$= coefficient of h^n on RHS

$= c_n c_0 e^{in\theta} + c_{n-1}c_1 e^{i(n-1)\theta}e^{-i\theta} + c_{n-2}c_2 e^{i(n-2)\theta}e^{-i2\theta} + \cdots + c_0 c_n e^{-in\theta}$

$= c_0 c_n[e^{in\theta} + e^{-in\theta}] + c_1 c_{n-1}[e^{i(n-2)\theta} + e^{-i(n-2)\theta}] + \cdots$

$$+\begin{cases} c_{\frac{n}{2}}^2 & n \text{ even} \\ c_{\frac{n-1}{2}} c_{\frac{n+1}{2}} 2(e^{i\theta}+e^{-i\theta}) & n \text{ odd} \end{cases}$$

$$= 2c_0c_n\cos n\theta + 2c_1c_{n-1}\cos(n-2)\theta + \cdots + \begin{cases} c_{\frac{n}{2}}^2 \\ c_{\frac{n-1}{2}} c_{\frac{n+1}{2}} 2\cos\theta \end{cases}$$

$\cos n\theta$ = polynomial in $\cos\theta$ of degree n, in alternate powers

$n, n-2, \cdots 0$ or 1 for odd and even.

Therefore $P_n(\mu)$ is a polynomial in μ of degree n, in alternate powers $n, n-2\cdots$

2) Putting $\mu = 1$ we have $\dfrac{1}{1-h} = \sum_0^\infty h^n P_n(1)$, therefore $P_n(1) = 1$

Since $P_n(-\mu) = (-1)^n P_n(\mu) \quad P_n(-1) = (-1)^n$

Values of P_0, P_1, P_2, P_3:

$$\frac{1}{[1+(h^2-2\mu h)]^{\frac{1}{2}}} = \sum_0^\infty c_r(2\mu h - h^2)^r$$

$= 1 + hc_1(2\mu - h) + h^2c_2(4\mu^2 - 4\mu h + h^2) + h^3c_3(8\mu^3 + \cdots)$

Therefore

$P_0(\mu) = 1$

$P_1(\mu) = 2c_1\mu = \mu$

$P_2(\mu) = -c_1 + 4c_2\mu^2 = -\frac{1}{2} + \frac{4.1.3}{2.4}\mu^2 = \frac{3\mu^2-1}{2}$

$P_3(\mu) = \frac{5}{2}\mu^3 - \frac{3}{2}\mu$

3) From $P_n(\mu) = 2c_0c_n\cos n\theta + \cdots$

$|P_n(\mu)| \le 2c_0c_n|\cos n\theta| + \cdots$

As $c_0c_1\cdots$ are all positive. If $-1 \le \mu \le 1 \quad \mu\cos\theta$ is real and $|\cos n\theta| \le 1$. Therefore $|P_n(\mu)| \le 2c_0c_n + \cdots = P_n(1) = 1$

4) Legendre's equation

We have the result that $r^n P_n(\cos\theta)$ is a solution of $\bigtriangledown^2 V = 0$ in spherical polar co-ordinates.

Therefore $\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\frac{\partial}{\partial r}r^n P_n(\cos\theta)\right\}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}r^n P_n(\cos\theta) = 0$

Therefore $n(n+1)r^{n-2}P_n(\mu) + r^{n+2}\frac{d}{d\mu}(1-\mu^2)\frac{d}{d\mu}P_n\mu = 0$

Therefore $P_n(\mu)$ satisfies $\frac{d}{d\mu}(1-\mu^2)\frac{dw}{d\mu} + n(n+1)w = 0$

[N.B. This equation has solutions linearly independent of $P_n(\mu)$ since it is of the second order. These solutions are unbounded at $\mu \pm 1$ corresponding to $\theta = 0$ or π (i.e. the 2-axis).

5) Orthogonal Property

i) $\frac{d}{d\mu}(1-\mu^2)\frac{d}{d\mu}P_n(\mu) + n(n+1)P_n(\mu) = 0$

ii) $\frac{d}{d\mu}(1-\mu^2)\frac{d}{d\mu}P_m(\mu) + m(m+1)P_m(\mu) = 0$

$(i)P_m(\mu) - (ii)P_n(\mu)$ gives

$$\frac{d}{d\mu}(1-\mu^2)\left\{P_m\frac{d}{d\mu}P_n - P_n\frac{d}{d\mu}P_m\right\} + [n(n+1) - m(m+1)]P_m P_n = 0$$

So $(n-m)(n+m+1)\int_{-1}^{1} P_m P_n d\mu + [(1-\mu^2)(P_m P_n' - P_n P_m')]_{-1}^{1} = 0$

Therefore $\int_{-1}^{1} P_m P_n d\mu = 0 \qquad m \neq n$

Value of $\int_{-1}^{1} P_n^2(\mu) d\mu$

$$\frac{1}{(1-2\mu h + h^2)} = \left[\sum_0^\infty h^n P_n(\mu)\right]^2$$

$$\int_{-1}^{1}\frac{d\mu}{1-2\mu h+h^2} = \int_{-1}^{1}\left[\sum_0^\infty h^n P_n(\mu)\right]^2 d\mu$$

LHS$=\left[-\frac{1}{2h}\log(1-2\mu h+h^2)\right]_{-1}^{1} = \frac{1}{2h}\log\frac{(1+h)^2}{(1-h)^2}$

$= \frac{1}{h}\log\frac{1+h}{1-h} = 2\sum_0^\infty \frac{h^{2n}}{2n+1}$

RHS$=\int_{-1}^{1}\sum_{n=0}^{\infty} h^n P_n(\mu)\sum_{m=0}^{\infty} h^m P_m(\mu)d\mu$

$= \sum_{n=0}^{\infty} h^n \int_{-1}^{1} P_n(\mu)\sum_{m=0}^{\infty} h^m P_m(\mu)d\mu$

$= \sum_{n=0}^{\infty} h^n \sum_{m=0}^{\infty} h^m \int_{-1}^{1} P_n(\mu)P_m(\mu)d\mu$

$= \sum_{n=0}^{\infty} h^{2n}\int_{-1}^{1} P_n^2(\mu)d\mu$ therefore $\int_{-1}^{1} P_n^2(\mu)d\mu = \frac{2}{2n+1}$

6) Rodriguez's Formula

P_m is perpendicular to P_n, $m \neq n$, $P_n(1) = 1$, $P_n(\mu)$ is of degree n.
Define $F(\mu)$ of degree $2n$ such that $F^{(n)}(\mu) = P_n(\mu)$

$F(1) = F'(1) = \cdots = F'(n-1) = 0.$

In fact $F(\mu) = \frac{1}{(n-1)!}\int_1^\mu (\mu-\lambda)^{n-1}P_n(\lambda)d\lambda$

i) $F(\mu)$ has a zero of order n at $\mu = +1$

ii) We show from the orthogonal properties that $F(\mu)$ has a zero of order n at $\mu = -1$.

Assuming this we have

$F(\mu) = (\mu-1)^n(\mu+1)^n *$ poly. of degree 0

$= c(\mu-1)^n(\mu+1)^n = c(\mu^2-1)^n$

Therefore $P_n(\mu) = c\frac{d^n}{d\mu^n}(\mu^2-1)^n$

To find c let $\mu = 1$

$1 = c\left[\frac{d^n}{d\mu^n}(\mu-1)^n(\mu+1)^n\right]_{\mu=1} = cn!2^n$

Therefore $c = \frac{1}{n!2^n}$ using Leibniz theorem.

Proof of (ii)

Since $P_0, P_1 \cdots$ are linearly independent polynomials, any polynomial $f(\mu)$ of degree r can be expressed uniquely as $c_0P_0 + c_1P_1 + \cdots + c_rP_r$.

P_n is perpendicular to P_r, $r < n$, therefore P_n is perpendicular to any polynomial of degree $r < n$. In particular P_n is perpendicular to $(1+\mu)^r$, $r < n$.

i.e. $\displaystyle\int_{-1}^{1} P_n(\mu)(1+\mu)^r d\mu = 0, \quad r < n$

i.e. $\displaystyle\int_{-1}^{1} F^{(n)}(\mu)(1+\mu)^r d\mu = 0, \quad r < n$

Denote this by $I_{n,r}$, where $I_{n,r} = 0$ for $0 < r < n$

$$I_{n,r} = -r\int_{-1}^{1} F^{(n-1)}(\mu)(1+\mu)^{r-1} d\mu$$

Therefore $I_{n-1,r-1} = 0$ for $1 < r < n$

By r integrations by parts $\displaystyle\int_{-1}^{1} F^{(n-r)}(\mu) d\mu = 0$

$[F^{(n-r-1)}(\mu)]_{-1}^{1} = 0 \quad F^{(n-r-1)}(1) = F^{(n-r-1)}(-1) = 0 \quad r = 0, 1, \cdots n-1$

Hence (ii) follows.

Suppose $f(\mu)$ has derivatives of all orders in $[-1, 1]$.

$$\int_{-1}^{1} f(\mu)P_n(\mu)d\mu = \frac{1}{2^n n!}\int_{-1}^{1} f(\mu)D^n(\mu-1)^n d\mu$$

$$= \frac{1}{2^n n!}\int_{-1}^{1}[-f'(\mu)]D^{n-1}(\mu^2-1)^n d\mu + 0$$

$$= \frac{(-1)^n}{2^n n!}\int_{-1}^{1} f^{(n)}(\mu)(\mu^2-1)d\mu = \frac{1}{2^n n!}\int_{-1}^{1}(1-\mu^2)^n f^{(n)}(\mu)d\mu$$

e.g. $\displaystyle\int_{-1}^{1} P_n^2(\mu) = \frac{1}{2^n n!}\int_{-1}^{1}(1-\mu^2)^n D^n P_n(\mu)d\mu = \frac{2}{2n+1}$

7) Recurrence Formulae

We have $\displaystyle G(\mu, h) \equiv (1 - 2\mu h + h^2)^{-\frac{1}{2}} = \sum_0^\infty h^n P_n(\mu)$

(a) $\dfrac{\partial G}{\partial h} = \dfrac{\mu - h}{(1 - 2\mu h + h^2)} G$

i.e. $(1 - 2\mu h + h^2) \sum_1^\infty n P_n(\mu) h^{n-1} = (\mu - h) \sum_0^\infty h^n P_n(\mu)$

Equating coefficients of h^n on each side

$$(n+1)P_{n+1}(\mu) - 2\mu n P_n(\mu) + (n-1)P_{n-1}(\mu) = \mu P_n(\mu) - P_{n-1}(\mu) \quad (i)$$

$$2P_2(\mu) - 2\mu P_1(\mu) = \mu P_1(\mu) - P_0(\mu) \quad (ii)$$

$$P_1(\mu) = \mu P_0(\mu) \quad (iii)$$

(i) gives $(m+1)P_{m+1}(\mu) - (2n+1)\mu P_n(\mu) + n P_{n-1}(\mu) = 0$

(b) $\dfrac{\partial g}{\partial \mu} = \dfrac{h}{(1 - 2\mu h + h^2)} G \qquad \dfrac{\partial G}{\partial h} = \dfrac{\mu - h}{(1 - 2\mu h + h^2)} G$

$$\left(\frac{1}{h} - h\right) \frac{\partial G}{\partial \mu} - 2h \frac{\partial G}{\partial h} = \frac{1 - h^2 - 2h(\mu - h)}{(1 - 2\mu h + h^2)} G = G$$

Therefore $\left(\dfrac{1}{h} - h\right) \dfrac{\partial G}{\partial \mu} = \left(2h \dfrac{\partial}{\partial h} + 1\right) G$

$$\left(\frac{1}{h} - h\right) \sum_1^\infty h^n P_n'(\mu) = \sum_0^\infty (2n+1) h^n P_n(\mu)$$

therefore $P_{n+1}'(\mu) - P_{n-1}'(\mu) = (2n+1)P_n(\mu)$

$P_2'(\mu) = 3P_1(\mu)$

This formula gives

$$\int_1^\mu P_n(\lambda) d\lambda = \frac{1}{2n+1} \{P_n + 1(\mu) - P_{n-1}(\mu)\}$$

(c) $\dfrac{\mu^2 - 1}{h} \dfrac{\partial G}{\partial \mu} - (\mu - h) \dfrac{\partial G}{\partial h} = -G$

Therefore $(\mu^2 - 1)P_n'(\mu) = n(\mu P_n(\mu) - P_{n-1}(\mu))$

$$= n \left\{ \frac{(n+1)P_{n+1}(\mu) + nP_{n-1}(\mu)}{2n+1} - P_{n-1}(\mu) \right\}$$

$$= \frac{n(n+1)}{2n+1} \{P_{n+1}(\mu) - P_{n-1}(\mu)\}$$

Differentiating the above gives

$$\frac{d}{d\mu}(\mu^2 - 1) \frac{dP_n(\mu)}{d\mu} = \frac{n(n+1)}{2n+1} \{P_{n+1}'(\mu) - P_{n-1}'(\mu)\} = n(n+1)P_n(\mu)$$

which is Legendre's equation.

[Note that $\dfrac{\partial}{\partial\mu}(\mu^2-1)\dfrac{\partial G}{\partial\mu} = h\dfrac{\partial}{\partial h}\left(h\dfrac{\partial}{\partial h}+1\right)G$ and this leads to the differential equation.]

Zeros of $P_n(\mu)$

$P_n(\mu) = \dfrac{1}{2^n n!}\dfrac{d^n}{d\mu^n}(\mu^2-1)^n$

$(\mu^2-1)^n$ has n zeros at -1 and n zeros at $+1$.

therfore $\dfrac{d}{d\mu}(\mu^2-1)^n$ has $n-1$ zeros at -1, $n-1$ zeros at $+1$ and therefore one (say μ) in $(-1,1)$, as it has $2n-1$ altogether.

Continuing this process $\dfrac{d^n}{d\mu^n}(\mu^2-1)^n$ has n zeros in $-1<\mu<1$, all simple.

Axially Symmetric Potentials (in spherical co-ordinates)

DIAGRAM

Let U be a solution of $\nabla^2 U = 0$, existing in $a \le r \le b$ and axially symmetric about Oz.

If (r,θ,ϕ) are spherical polar co-ordinates, where $\theta = 0$ and $\theta = \pi$ is the same z-axis, then U has the form $\sum_0^\infty \left(A_n r^n + \dfrac{B_n}{r^{n+1}}\right) P_n(\cos\theta)$

on $\theta = 0$ this becomes $U(r,0) = \sum_0^\infty \left(A_n r^n + \dfrac{B_n}{r^{n+1}}\right)$.

Conversely if $U(r,0)$ has this form and exists in $a \le r \le b$ then

$U(r,\theta) = \sum_0^\infty \left(A_n r^n + \dfrac{B_n}{r^{n+1}}\right) P_n(\cos\theta)$

Example

DIAGRAM

$U = \displaystyle\int \frac{dS}{|\vec{r}-\vec{r_0}|}$

where $\vec{r}$ is the position vector of the field point, and where $\vec{r_0}$ is the position vector of a point on the disc, and the integral is taken over the disc with boundary $r = C,\ \theta = \alpha$, referred to Oz.

DIAGRAM

$U(r,0) = 2\pi \int_)^a \frac{pdp}{(p^2+h^2)^{\frac{1}{2}}} = 2\pi[\sqrt{a^2+h^2} - |h|] = 2\pi(R-h)$

$h = c\cos\alpha - r$

Therefore $U(r,0) = 2\pi[(r^2+c^2-2rc\cos\alpha)^{\frac{1}{2}} - (c\cos\alpha - r)]$

$(r^2+c^2-2rc\cos\alpha)^{\frac{1}{2}} = (r^2+c^2-2rc\cos\alpha)(r^2+c^2-2rc\cos\alpha)^{\frac{1}{2}}$

$$= (r^2+c^2-2rc\cos\alpha)\begin{cases} \sum_{n=0}^{\infty} \frac{c^n}{r^{n+1}} P_n(\cos\alpha) & r < c \\ \sum_{n=0}^{\infty} \frac{r^n}{c^{n+1}} P_n(\cos\alpha) & r > c \end{cases}$$

For $r > c$, putting $\lambda = \cos\alpha$,

$$\frac{(r^2+c^2-2rc\cos\alpha)^{\frac{1}{2}}}{r} = \left(1 + \frac{c^2}{r^2} - 2\lambda\frac{e}{r}\right)\sum_0^{\infty}\left(\frac{c}{r}\right)^n P_n(\lambda)$$

$$= P_0(\lambda) + \frac{c}{r}\{P_1(\lambda) - 2\lambda P_0(\lambda)\} + \sum_{r=2}^{\infty}\left(\frac{c}{r}\right)^n \{P_n(\lambda) - 2\lambda P_{n-1}(\lambda) + P_{n-2}(\lambda)\}$$

Therefore $(r^2+c^2-2rc\cos\alpha)^{\frac{1}{2}} - (r - c\lambda)$

$$= r\left(1 + \frac{c}{R}(-\lambda) + \sum_{n=2}^{\infty}\{P_n(\lambda) - 2\lambda P_{n-1}(\lambda) + P_{n-2}(\lambda)\}\right) - (r - c\lambda)$$

$(r > c \Rightarrow r > c\lambda)$

$$= r\sum_{n=2}^{\infty}\frac{c^n}{r^n}F_n(\lambda), \qquad F_n(\lambda) = P_n(\lambda) - 2\lambda P_{n-1}(\lambda) + P_{n-2}(\lambda)$$

Therefore $\frac{U(r,0)}{2\pi} = \sum_{n=2}^{\infty}\frac{c^n}{r^{n-1}}F_n(\lambda) = \sum_{n=0}^{\infty}\frac{c^{n+2}}{r^{n+1}}F_{n+2}(\lambda) \qquad r > c$

Therefore $\frac{U(r,0)}{2\pi} = \sum_{n=0}^{\infty}\frac{c^{n+2}}{r^{n+1}}P_n(\cos\theta)F_{n+2}(\cos\alpha)$

[For large r, RHS$\approx \frac{c^2}{r}F_2(\cos\alpha)$

$F_2(\cos\alpha) = P_2(\cos\alpha) - 2\cos\alpha P_1(\cos\alpha) + P_0(\cos\alpha)$

$$= \frac{3\cos^2\alpha - 1}{2} - 2\cos^2\alpha + 1 = \frac{1}{2} - \frac{1}{2}\cos^2\alpha = \frac{1}{2}\sin^2\alpha$$

Therefore $U(r,\theta) \sim \frac{\pi c^2\sin^2\alpha}{r} = \frac{\pi a^2}{r}$ as $r \to \infty$]

$F_{n+2}(\lambda) = P_{n+2} - 2\lambda P_{n+1} + P_n$

$$= P_{n+2} + P_n - 2\left\{\frac{(n+2)P_{n+2} + (n+1)P_n}{2n+3}\right\}$$

$$= \frac{-P_{n+2} + P_n}{2n+3} = \frac{(1-\lambda^2)P'_{n+1}(\lambda)}{(n+1)(n+2)}$$

Example of Boundary Problem
To find a potential V existing in $0 \le r \le a$ such that $V + U = 0$ on $r = a$ where $U(r,\theta)$ is the potential considered above.
$V(r,\theta)$ must be of the form $\sum_{n=0}^{\infty} \frac{r^n}{a^n} A_n P_n(\cos\theta)$
Hence we require $\sum_0^{\infty} A_n P_n(\cos\theta) + U(a,\theta) = 0$

$$\frac{U(a,\theta)}{2\pi} = \sum_{n=0}^{\infty} \frac{c^{n+2}}{a^{n+1}} P_n(\cos\theta) F_{n+2}(\lambda)$$

Therefore $\frac{V}{2\pi} = -\sum_0^{\infty} r^n \frac{c^{n+2}}{a^{2n+1}} P_n(\cos\theta) F_{n+2}(\lambda)$.

Definition - Solid Harmonic of degree n
If $f(x,y,z)$ is a polynomial in x,y,z homogeneous and of degree n, and if $\bigtriangledown^2 f = 0$, then f is said to be a solid harmonic of degree n.
Example: 1; x, y, z; $yz, zx, xy, z^2 - x^2, z^2 - y^2$, etc.

Definition - Surface Harmonic of degree n
If $f(x,y,z) = r^n S_n(\mathbf{u})$ or $r^n S_n(\theta,\phi)$ where S_n depends only on the unit vector $\mathbf{u}$ along the position vector, or on the spherical polar angles θ, ϕ, S_n is called a surface harmonic of degree n.

Differential Equation satisfied by $\mathbf{S_n}$
Substitute $f = r^n S_n$ in $\bigtriangledown^2 f = 0$.

$$\left\{ \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{d}{d\theta} + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right\} f = 0$$

Therefore $\left\{ \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{d}{d\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} + n(n+1) \right\} S_n = 0 \qquad (1)$

This equation admits solutions of the form $s(\theta)e^{\pm im\phi}$, m constant, where

$$\frac{1}{\sin\theta} \frac{d}{d\theta} \sin\theta \frac{ds}{d\theta} + \left(n(n+1) - \frac{m^2}{\sin^2\theta} \right) s = 0$$

Putting $\cos\theta = \mu$, so $\frac{1}{\sin\theta} \frac{d}{d\theta} = -\frac{d}{d\mu}$ this becomes

$$\frac{d}{d\mu}(1-\mu^2)\frac{ds}{d\mu} + \left(n(n-1) - \frac{m^2}{1-\mu^2} \right) s = 0 \qquad (2)$$

Equation (2) is called Legendre's associated equation. For $m = 0$ it reduces to Legendre's equation. In this case S_n is independent of ϕ, i.e. is axially symmetric. One solution is $P_n(\mu)$.

The number of linearly independent Surface Harmonics of degree n is $\mathbf{2n+1}$

f can always be written

$$f = \phi_n(x,y) + \frac{z}{1!}\phi_{n-1}(x,y) + \frac{z^2}{2!}\phi_{n-2}(x,y) + \cdots$$

where ϕ_r is a homogeneous polynomial in x, y of degree r.

$$\nabla^2 f = \nabla_1^2 + \frac{\partial^2}{\partial z^2} = (\nabla_1^2\phi_n + \phi_{n-2}) + \frac{z}{1!}(\nabla_1^2\phi_{n-1} + \phi_{n-3})$$
$$+ \cdots + \frac{z^{n-3}}{(n-3)!}(\nabla_1^2\phi_3 + \phi_1) + \frac{z^{n-2}}{(n-2)!}(\nabla_1^2\phi_2 + \phi_0)$$

Since this must vanish identically

$\nabla_1^2\phi_n + \phi_{n-2} = 0 \qquad \nabla_1^2\,\phi_{n-1} + \phi_{n-3} = 0$

$\nabla_1^2\phi_{n-2} + \phi_{n-4} = 0\cdots \qquad \nabla_1^2\,\phi_{n-3} + \phi_{n-5} = 0\cdots$

Therefore $\phi_n,\ \phi_{n-1}$ are arbitrary polynomials in x, y of degrees n and $n-1$, and for the others we have

$\phi_{n-2r} = (-1)^r(\nabla_1^2)^r\phi_n$

$\phi_{n-2r-1} = (-1)^r(\nabla_1^2)^r\phi_{n-1}$

Therefore

$$f = \phi_n - \frac{z^2}{2!}\nabla_1^2\phi_n + \frac{z^4}{4!}(\nabla_1^2)^2\phi_n - \cdots + \frac{z}{1!}\phi_{n-1} - \frac{z^3}{3!}(\nabla_1^2)\phi_{n-1} + \frac{z^5}{5!}(\nabla_1^2)^2\phi_{n-1} - \cdots$$

where both series must terminate.

ϕ_n can have any one of the forms $x^n, x^{n-1}y, \cdots y^n$. There are $n+1$ of these, and they are linearly independent.

ϕ_{n-1} can have any one of the forms $x^{n-1}, x^{n-2}y, \cdots y^{n-1}$. There are n of these, and they are linearly independent.

Therefore the total number of forms is $2n+1$ and the corresponding f's are linearly independent.

Associated Legendre Functions [Ferrer's definition]

$$P_n^m(\mu) = (1-\mu^2)^{\frac{m}{2}}\frac{d^m}{d\mu^m}P_n(\mu) = (1-\mu^2)^{\frac{m}{2}}\frac{1}{2^n n!}\frac{d^{m+n}}{d\mu^{m+n}}(\mu^2-1)^n$$

is the associated Legendre function of the first kind of degree n, order m. There are $n+1$ such functions for $m = 0, 1, 2\cdots n$.

We show

i) $r^n e^{\pm mi\phi}P_n^m(\cos\theta) =$ polynomial in x, y, z of degree n.

ii) $\bigtriangledown^2 r^n e^{\pm mi\phi} P_n^m(\cos\theta) = 0$

i) $P_n^m(\cos\theta) = \sin^m\theta$ [Poly. in $\cos\theta; \cos^{n-m}\theta \cdots \begin{cases} \cos\theta & n-m \text{ odd} \\ 1 & n-m \text{ even} \end{cases}$

$r^n e^{im\phi} P_n^m \cos\theta = (r\sin\theta)^m e^{im\phi} r^{n-m}[\cdots]$

$= (x+iy)^m$ [Poly in $z, r^2; z^{n-m}, z^{n-m-2}r^2 \cdots \begin{cases} zr^{n-m-1} & n-m \text{ odd} \\ r^{n-m} & n-m \text{ even} \end{cases}$

$=$ poly in $x, y, z, \sin\theta,\ \ r^2 = x^2 + y^2 + z^2$

ii) To show $\bigtriangledown^2 r^n e^{\pm im\phi}(1-\mu^2)^{\frac{m}{2}} \dfrac{d^m}{d\mu^m} P_n(\mu) = 0$

This is so if $e^{\pm im\phi}(1-\mu^2)^{\frac{m}{2}} \dfrac{d^m}{d\mu^m} P_n(\mu)$

satisfies $\left[\dfrac{\partial}{\partial\mu}(1-\mu^2)\dfrac{\partial}{\partial\mu} + \left(n(n+1) + \dfrac{1}{1-\mu^2}\right)\dfrac{ap^2}{\partial\phi^2}\right](\ ) = 0$

i.e. if $(1-\mu^2)^{\frac{m}{2}} D^m P_n(\mu)$ satisfies

$$L(m; w) = \left[\frac{d}{d\mu}(1-\mu^2)\frac{d}{d\mu} + n(n+1) - \frac{m^2}{1-\mu^2}\right] w = 0$$

This is know as Legendre's Associated equation.

Now

$$L(m : (1-\mu^2)^{\frac{m}{2}} W) = \frac{d}{d\mu}\left\{(1-\mu^2)^{\frac{m}{2}+1}\frac{dW}{d\mu} - m\mu(1-\mu^2)^{\frac{m}{2}} W\right\}$$

$$+ \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}(1-\mu^2)^{\frac{m}{2}} W$$

$$= (1-\mu^2)^{\frac{m}{2}+1}\frac{d^2W}{d\mu^2} + (-\mu(m+2) - m\mu)(1-\mu^2)^{\frac{m}{2}}\frac{dW}{d\mu}$$

$$-mW((1-\mu^2)^{\frac{m}{2}} - m\mu^2(1-\mu^2)^{\frac{m}{2}-1})$$

$$+ \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}(1-\mu^2)^{\frac{m}{2}} W$$

$$= (1-\mu^2)^{\frac{m}{2}} L_1(m : W)$$

where $L_1(m:W)$

$$= \left\{(1-\mu^2)\frac{d^2}{d\mu^2} - 2(m+1)|mu\frac{d}{d\mu} + n(n+1) - m(m+1)\right\} W$$

We must now show that

$L_1(m : D^m P_n(\mu)) = 0$

Since $L_1(m; W) = 0 \Rightarrow DL_1(m : W) = 0$ we get

$[(1-\mu^2)D^3 + (-2\mu D^2 - 2(m+1)\mu D^2)$

$\quad +n(n+1) - m(m+1) - 2(m+1)D]W = 0$

i.e. $[(1-\mu^2)D^2 - 2\mu(m+2)D + n(n+1) - (m+1)(m+2)]DW = 0$

i.e. $L_1(m+1; DW) = 0$

i.e. $L_1(m; W) = 0 \Rightarrow L_1(m+1; DW) = 0$

i.e. $L_1(0, W) = 0 \Rightarrow L_1(m; D^m W) = 0$

$L_1(0; P_n(\mu)) = [(1-\mu^2)D^2 - 2\mu D + n(n+1)]P_n(\mu) = 0$

Therefore $L_1(m; D^m P_n(\mu)) = 0$ as required.

General Surface Harmonic of degree n

Giving m the values $0, 1, \cdots n$ in $r^n e^{\pm mi\phi} P_n^m(\cos\theta)$ we have $r^n P_n(\cos\theta)$, $r^n e^{\pm i\phi} P_n'(\cos\theta), \cdots r^n e^{\pm ni\phi} P_n^n(\cos\theta)$.

These are $2n+1$ in number and are linearly independent (from the orthogonality of 1, $e^{\pm i\phi} \cdots$ over $0 \leq \phi \leq 2\pi$).

Therefore

$$S_n = A_0 P_n(\mu) + \sum_{m=1}^{n} (C_m e^{mi\phi} + C_m' e^{-mi\phi}) P_n^m(\mu)$$

$$= A_0 P_n(\mu) + \sum_{m=1}^{n} (A_m \cos m\phi + B_m \sin m\phi) P_n^m(\mu)$$

Solutions of Legendre's equation when n $\neq$ integer