The Gamma Function

I. Integral Definition

$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$

This is convergent since $t^m e^{-t} < t^{m-n} n!$ thus proving convergence at the upper limit and also at the lower limit if $z > 0$ since $\lim_{\epsilon \to 0} \int_\epsilon t^{z-1} dt$ exists.

For z complex the convergence holds for $R(z) > 0$.

We have

i) $\Gamma(1) = \int_0^\infty e^{-t} dt = 1$

ii) $z\Gamma(z) = \Gamma(z+1)$

$$\int_0^\infty t^{z-1} e^{-t} dt = \left[\frac{t^z}{z} e^{-t} \right]_0^\infty + \int_0^\infty \frac{t^z}{z} e^{-t} dt$$

$$\Gamma(z) = 0 + \frac{1}{z} \Gamma(z+1)$$

iii) $\Gamma(n+1) = n!$ $\qquad$ ($n \geq 0$, an integer)

II. Alternative Integral Definition

Substitute $t = u^2$, this gives $\Gamma(z) = 2 \int_0^\infty u^{2z-1} e^{-u^2} du$

III. Limit Definition (Euler)

$$\Gamma(z) = \lim_{n \to \infty} \frac{n! n^z}{z(z+1) \cdots (z+n)}$$

This holds for all z except $0, -1, -2, \cdots$

We can derive III from I in two ways

a) $e^{-t} = \lim_{n \to \infty} \left(1 - \frac{t}{n} \right)^n$

Define $\Gamma_n(z) = \int_0^n t^{z-1} \left(1 - \frac{t}{n} \right)^n dt$

$$\int_0^n t^{z-1} e^{-t} dt - \Gamma_n(z) = \int_0^n t^{z-1} \left\{ e^{-t} - \left(1 - \frac{t}{n} \right)^n \right\} dt$$

$$\int_0^n t^{z-1} e^{-t} \left\{ 1 - e^t \left(1 - \frac{t}{n} \right)^n \right\} dt$$

We have

i) $e^t \geq t + 1$ for all t

ii) $e^{-t} \geq 1 - t$ for all t

From ii) $1 \geq e^t(1-t)$ for all t

From i) multiplying by $(1-t)$, $t \leq 1$

$e^t(1-t) \geq (1-t^2)$

Therefore $1 \geq e^t(1-t) \geq 1 - t^2$ for $t \leq 1$

Taking the nth power $(0 \leq t \leq 1)$

$1 \geq e^{nt}(1-t)^n \geq (1-t^2)^n$

Replace t by $\frac{t}{n}$

$$1 \geq e^t \left(1 - \frac{t}{n}\right)^n \geq \left(1 - \frac{t^2}{n^2}\right)^n \qquad 0 \leq t \leq n$$

Hence $1 - \left(1 - \dfrac{t^2}{n^2}\right)^n \geq 1 - e^t \left(1 - \dfrac{t}{n}\right)^n \geq 0$

For $0 \leq x \leq 1$ $\quad 0 \leq 1 - x^n \leq n(1-x)$

Therefore $\dfrac{t^2}{n} \geq 1 - e^t \left(1 - \dfrac{t}{n}\right)^n \geq 0$

Therefore $\left|\displaystyle\int_0^n t^{z-1} e^{-t} \left(1 - e^t \left(1 - \frac{t}{n}\right)^n\right) dt\right|$

$$\leq \int_0^n |t^{z-1}| e^{-t} \left|1 - e^t \left(1 - \frac{t}{n}\right)^n\right| dt$$

$$\leq \int_0^n |t^{z-1}| e^{-t} \frac{t^2}{n} dt$$

$|t^{z-1}| = e^{(x-1)\log t} = t^{x-1} = t^{Re(z)-1}$

Therefore $\left|\displaystyle\int_0^n t^{z-1} e^{-t} \left(1 - e^t \left(1 - \frac{t}{n}\right)^n\right) dt\right| \leq \dfrac{1}{n} \displaystyle\int_0^n t^{Re(z)+1} e^{-t} dt$

$$\leq \frac{1}{n} \int_0^\infty t^{Re(z+1)} e^{-t} dt = \frac{\Gamma(Re(z+2))}{n} = \frac{const}{n}$$

Hence $\displaystyle\lim_{n\to\infty} \int_0^n t^{z-1} e^{-t} \left(1 - e^t \left(1 - \frac{t}{n}\right)^n\right) dt = 0$

Hence $\lim_{n\to\infty}\left\{\int_0^n t^{z-1}e^{-t}dt - \Gamma_n(z)\right\} = 0$

Therefore $\Gamma(z) = \lim_{n\to\infty}\Gamma_n(z)$

$$\begin{aligned}\Gamma_n(z) &= \int_0^n t^{z-1}\left(1-\frac{t}{n}\right)^n dt \\ &= n^z\int_0^1 s^{z-1}(1-s)^n ds \\ &= n^z\frac{n}{z}\frac{n-1}{z+1}\cdots\frac{1}{z+n}.1\end{aligned}$$

Therefore $\Gamma(z) = \lim_{n\to\infty}\dfrac{n!n^z}{z(z+1)\cdots(z+n)}$

b) Define $f_n(t) = \begin{cases}(1-\frac{t}{n})^n t^{z-1} & 0 < t < n \\ 0 & t \geq n\end{cases}$

then $0 \leq |f_n(t)| \leq |e^{-t}t^{z-1}|$ for every n, and $f_n(t) \to e^{-t}t^{z-1}$ as $n \to \infty$ for every t, also $\exists \int_0^\infty e^{-t}t^{z-1}dt < \infty$

Hence by Lebesgue's theorem on dominated cgce.

$\int_0^\infty \lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\int_0^\infty f_n(t)dt$

i.e. $\int_0^\infty e^{-t}t^{z-1}dt = \lim_{n\to\infty}\int_o^n f_n(t)dt$

and the result follows as before.

IV. Infinite Product Definition (Weierstrass)

Infinite Products

We have a sequence $1+a_1$, $1+a_2$, $\cdots$ none of which are zero. We form the product defined by

$\prod_m = (1+a_1)(1+a_2)\cdots(1+a_m)$

If $\prod_m$ tends to a limit other than zero as $m \to \infty$ then the infinite product $(1+a_1)(1+a_2)\cdots$ is said to converge and is written $\prod_{n=1}^{\infty}(1+a_n)$.

A necessary condition for convergence is $\lim n \to \infty a_n = 0$, for $1+a_n = \dfrac{\prod_n}{\prod_{n-1}}$ and we have $\lim n \to \infty \prod_n = \lim_{n\to\infty} \prod_{n-1} \neq 0$

A sufficient condition for convergence is that the series $\sum_{n=1}^{\infty} \log(1+a_n)$ is convergent.
(Here we take the principal value of the log.
i.e. such that $-\pi < \arg(1+a_n) < \pi$ and $\log(1+a_n) \to 0$ as $n \to \infty$ and $a_n \to 0$.)

$$\text{for } \prod_m = \exp\left\{\log \prod_1^m (1+a_n)\right\}$$
$$= \exp\left\{\sum_{n+1}^{m} \log(1+a_n)\right\}$$

Hence, since the exponential function is continuous, $\lim \exp s_m = \exp \lim s_m$, and the result follows when we take $s_m = \sum_{n=1}^{m} \log(1+a_n)$.
If $\sum \log(1+a_n)$ is absolutely convergent then $\prod(1+a_n)$ is said to be absolutely convergent.

Theorem
A necessary and sufficient condition for absolute convergence is that the series $\sum a_n$ is absolutely convergent.
Proof
Since $\lim_{n\to\infty} a_n = 0$ we can find m where $|a_n| < \frac{1}{2}$ for $n \geq m$.
Then

$$\begin{aligned}
\log(1+a_n) &= a_n - \frac{a_n^2}{2} + \frac{a_n^3}{3} - \cdots \\
\frac{\log(1+a_n)}{a_n} - 1 &= -\frac{a_n}{2} + \frac{a_n^2}{3} - \cdots \\
\left|\frac{\log(1+a_n)}{a_n} - 1\right| &\leq \frac{|a_n|}{2} + \frac{|a_n|^2}{3} + \cdots \\
&\leq \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots \qquad n \geq m \\
&\leq \frac{1}{2}
\end{aligned}$$

Therefore $\frac{1}{2} \leq \left|\frac{\log(1+a_n)}{a_n}\right| \leq \frac{3}{2}$

Therefore $\frac{1}{2}|a_n| \leq |\log(1+a_n)| \leq \frac{3}{2}|a_n| \qquad n \geq m$

Hence by the comparison test $\sum |\log(1+a_n)|$ converges or diverges as $\sum |a_n|$ converges or diverges.

N.B. If a finite number of factors $(1+a_1), \cdots$ vanish and if the product omitting these factors is convergent the product is said to converge to zero. If no factor vanishes but $\lim_{m\to\infty} \prod_m = 0$ then the product is said to diverge to zero.

Returning to the Gamma-Function we have:

$$\frac{1}{\Gamma(z)} = \lim_{n\to\infty} zn^z \left(1+\frac{z}{1}\right)\left(1+\frac{z}{2}\right)\cdots\left(1+\frac{z}{n}\right)$$

The product $\prod \left(1+\frac{z}{n}\right)$ is divergent $(z \neq 0)$ for

$$\log\left(1+\frac{z}{n}\right) = \frac{z}{n} + O\left(\frac{z^2}{n^2}\right) \qquad \left\{\frac{|z|}{n} < 1\right\}$$

and the series $\sum \frac{1}{n}$ is divergent and $\sum O(\frac{1}{n^2})$ is convergent.

Hence $\sum \log(1+a_n)$ is divergent $(z \neq 0)$.

Now $\prod_{n=1}^{m}\left(1+\frac{z}{n}\right) = \prod_{n=1}^{m}\left\{\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right\} e^{z\left(1+\frac{1}{2}+\cdots+\frac{1}{m}\right)}$

Also $\log\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}} = O\left(\frac{z^2}{n^2}\right)$

Hence the product $\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}$ is convergent.

$$\text{Therefore } \frac{1}{\Gamma(z)} = z\lim_{m\to\infty} m^{-z}e^{z\left(1+\frac{1}{2}+\cdots+\frac{1}{m}\right)}\prod_{n=1}^{m}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}$$
$$= z\lim_{m\to\infty} e^{z\left(1+\frac{1}{2}+\cdots+\frac{1}{m}-\log m\right)}\prod_{n=1}^{m}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}$$

Now $\lim_{m\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{m} - logm\right) = \gamma$ (Euler's constant)

Therefore $\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}$

This is Weierstrass' definition.

Euler's Constant - Proof that γ exists.

$$1+\frac{1}{2}+\cdots+\frac{1}{m}-\log(m+1)=\sum_{n=1}^{m}\frac{1}{n}-\sum_{n=1}^{m}\log\frac{n+1}{n}$$

$$=\sum_{n=1}^{m}u_n \text{ where } u_n=\tfrac{1}{n}-\log\tfrac{n+1}{n}$$

$$=\frac{1}{n}\int_0^1 dt-\int_0^1\frac{dt}{t+n}=\int_0^1\left(\frac{1}{n}-\frac{1}{t+n}\right)dt$$

$$=\int_0^1\frac{tdt}{n(t+n)}\leq\int_0^1\frac{t}{n^2}dt=\frac{1}{2n^2}$$

Therefore $\sum u_n$ is convergent by comparison with $\sum\frac{1}{2n^2}$

Therefore $\lim_{m\to\infty}\left[1+\frac{1}{2}+\cdots+\frac{1}{m}-\log(m+1)\right]=\sum_{n=1}^{\infty}u_n$

Now $1+\frac{1}{2}+\cdots+\frac{1}{m}-\log m=\left[1+\frac{1}{2}\cdots\frac{1}{m}-\log(m+1)\right]+\log\frac{m+1}{m}$

and $\lim_{m\to\infty}\log\frac{m+1}{m}=\log 1=0$. Therefore $\gamma=\sum_{n=1}^{\infty}u_n$.

Properties of $\Gamma(z)$ (Weierstrass Form)

i) The RHS is convergent for all $z<\infty$. Hence $\Gamma(z)$ has no zeros.

ii) The RHS has simple zeros at $z=0,-1,\cdots$. Hence $\Gamma(z)$ has simple poles at these points.

iii) $z\Gamma(z)=\Gamma(z+1)$

iv) $\Gamma(z)\Gamma(1-z)=\pi\csc\pi z$ $\qquad\left(\frac{\sin z}{z}=\prod\left(1-\frac{z^2}{n^2\pi^2}\right)\right)$

v) $2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})=\Gamma(\frac{1}{2})\Gamma(2z)$ duplication formula

vi) $\Gamma(\frac{1}{2})=\pi^{\frac{1}{2}}$ (same as iv) but with $z=\frac{1}{2}$)

Behaviour of $\Gamma(x)$ for real x

$\Gamma(x)>0$ for $x>0$ $\qquad\Gamma(n)=(n-1)!$

$z(z+1)\cdots(z+n)\Gamma(z)=\Gamma(n+1+z)$

Therefore $(z+n)\Gamma(z)=\dfrac{\Gamma(n+1+z)}{z(z+1)\cdots(z+n+1)}$

$$\lim_{z\to -n}(z+n)\Gamma(z) = \frac{\Gamma(1)}{-n(-n+1)\cdots(-1)} = \frac{(-1)^n}{n!}$$
This is the residue at the simple pole $z=-n$.
DIAGRAM

The Beta Function

$$B(m,n) = \int_0^1 u^{m-1}(1-u)^{n-1}du \qquad (m,n>0)$$

We shall show that $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

We note that $B(m,n) = B(n,m)$ (putting $u = 1-v$).
Also putting $u=\cos^2\theta$

$$B(m,n) = 2\int_0^{\frac{\pi}{2}} \cos^{2m-1}\theta \sin^{2n-1}\theta d\theta$$

Define $\Gamma(m;R) = 2\int_0^R x^{2m-1}e^{-x^2}dx$

then $\lim_{R\to\infty}\Gamma(m;R) = \Gamma(m)$

$$\Gamma(m;R)\Gamma(n;R) = 4\int_0^R x^{2m-1}e^{-x^2}dx\int_0^R y^{2n-1}e^{-y^2}dy$$

Assume for the moment that $m,n\geq \frac{1}{2}$. Then $x^{2m-1}e^{-x^2}y^{2n-1}e^{-y^2}$ is a continuous function of x and y in $x\geq 0,\ \ y\geq 0$.

Hence $4\displaystyle\iint_{\substack{\text{square}\\ 0\leq x\leq R\\ 0\leq y\leq R}} x^{2m-1}e^{-x^2}y^{2n-1}e^{-y^2}dxdy = 4\int_0^R x^{2m-1}e^{-x^2}dx\int_0^R y^{2n-1}e^{-y^2}dy$

DIAGRAM

$$4\iint_{\text{square}} = 4\iint_{\substack{\text{quadrant}\\ 0\leq\theta\leq\frac{\pi}{2}\\ 0\leq r\leq R}} + 4\iint_{\sum}$$

$$4\iint_{\text{quadrant}} = 4\iint x^{2m-1}y^{2n-1}e^{-r^2}rdrd\theta$$
$$= 4\int_0^R r^{2m+2n-1}e^{-r^2}dr\int_0^{\frac{\pi}{2}}\cos^{2m-1}\theta\sin^{2n-1}\theta d\theta$$
$$= \Gamma(m+n;R)B(m,n)$$

Therefore $\Gamma(m;R)\Gamma(n;R) = \Gamma(m+n;R)B(m,n) + 4\iint\limits_{\Sigma}$

$$\left|4\iint\limits_{\Sigma}\right| \leq 4\iint\limits_{\Sigma} x^{2m-1}y^{2n-1}e^{-x^2-y^2}dxdy \qquad (m, n \text{ are real})$$

$$\leq 4\iint\limits_{\substack{0\leq\theta\leq\frac{\pi}{2} \\ R\leq r\leq R\sqrt{2}}} r^{2m+2n-1}e^{-r^2}\cos^{2m-1}\theta\sin^{2n-1}\theta drd\theta$$

$$= 4\int_R^{R\sqrt{2}} r^{2m+2n-1}e^{-r^2}dr\int_0^{\frac{\pi}{2}}\cos^{2m-1}\theta\sin^{2n-1}\theta d\theta$$

$$= [\Gamma(m+n;R\sqrt{2}) - \Gamma(m+n;R)]B(m,n)$$

Therefore $\lim\limits_{R} \to \infty \iint\limits_{\Sigma} = 0$

Therefore as $R \to \infty \qquad \Gamma(m)\Gamma(n) = \Gamma(m+n)B(m,n)$

This proof holds for $m, n \geq \frac{1}{2}$.

Extension to $m, n > 0$

We have, for $m > 0$, $\Gamma(m) = \int_0^\infty t^{m-1}e^{-t}dt$ where the integral exists as an improper integral when $m > 0$.

Also $m\Gamma(m) = \Gamma(m+1) \qquad (m > 0)$

$$B(m,n) = \lim_{\alpha\to 0+,\,\beta\to 0-}\int_\alpha^{1-\beta} u^{m-1}(1-u)^{n-1}du$$

$$B(m,n+1) = \lim_{\alpha\to 0+,\,\beta\to 0-}\int_\alpha^{1-\beta} u^{m-1}(1-u)^{n}du$$

$$= \lim_{\alpha,\beta\to 0}\left\{\left[\frac{u^m}{m}(1-u)^n\right]_\alpha^{1-\beta} + \frac{n}{m}\int_\alpha^{1-\beta} u^m(1-u)^{n-1}du\right\}$$

$$= \lim_{\alpha,\beta\to 0}\left[\frac{-\alpha^m(1-\alpha)^n + (1-\beta)^m\beta^n}{m}\right] + \frac{n}{m}B(m+1,n)$$

Therefore $\dfrac{B(m,n+1)}{n} = \dfrac{B(m+1,n)}{m} \qquad (i)$

$$\int_0^1 u^{m-1}(1-u)^{n-1} = \int_0^1 u^{m-1}(1-u)^{n-1}\{u + (1-u)\}du$$

Therefore $B(m,n) = B(m+1,n) + B(m,n+1) \qquad (ii)$

Hence from (i) and (ii)

$$\frac{B(m,n+1)}{n} = \frac{B(m+1,n)}{m} = \frac{B(m+1,n)+B(m,n+1)}{m+n} = \frac{B(m,n)}{m+n}$$

Therefore $B(m+1,n) = \dfrac{m}{m+n}B(m,n)$

$B(m,n+1) = \dfrac{n}{m+n}B(m,n)$

Therefore

$$B(m+1,n+1) = \frac{n}{m+n+1}B(m+1,n) = \frac{mn}{(m+n)(m+n+1)}B(m,n)$$

Therefore $B(m,n) = \dfrac{(m+n)(m+n+1)}{mn}\dfrac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \qquad (m,n>0)$$

Example on Fubini's Theorem

$\Gamma(x)\Gamma(y) = \int_0^\infty e^{-t}t^{x-1}dt \int_0^\infty e^{-u}u^{y-1}du$

$= \int_0^\infty t^{x-1}dt \int_0^\infty e^{-(t+u)}u^{y-1}du$

(Using Fubini's theorem and regarding the integral as a multiple integral.)

$\int_0^\infty t^{x-1}dt \int_t^\infty e^{-u}(u-t)^{y-1}du$

$= \int_0^\infty t^{x-1}dt \int_0^\infty e^{-u}(u-t)^{y-1}X(ut)dt$

(Note $X(ut) = 1$ if $u > t$, but $X(ut) = 0$ if $u \le t$)

$= \int_0^\infty e^{-u}du \int_0^u t^{x-1}(u-t)^{y-1}dt$ (Fubini's theorem)

$= \int_0^\infty e^{-u}du \int_0^1 u^{x+y-1}w^{x-1}(1-w)^{y-1}dw$ $(t = uw)$

$= \int_0^\infty e^{-u}u^{x+y-1}du \int_0^1 w^{x-1}(1-w)^{y-1}dw$

$= \Gamma(x+y)B(x;y)$

(All valid for $x > 0,\ y > 0$)

Contour Integral

DIAGRAM

Consider $\int t^{z-1}e^{-t}dt$ around the contour $ABA'C'DCA$ where CDC' is the circle $|t| = \epsilon$ and AC, $C'A'$ are the upper and lower sides of the real axis from $t = \epsilon$ to $t = R$, and ABA' is a simple loop. If z is not an integer $t^{z-1} = e^{(z-1)\log t}$ is not one-valued (as a function of t). Choose that branch of $\log t$ which is real when t is at A. Hence along ABA' $\arg t$ increases from 0 to 2π; along $A'C'$ $\arg t = 2\pi$; along $C'DC$ $\arg t$ decreases from 2π to 0; along CA $\arg t = 0$ i.e. $\log t$ returns to its initial value ($\log R$) and hence so does

t^{z-1}, and also $t^{z-1}e^{-t}$ (since e^{-t} is one-valued). Hence $t^{z-1}e^{-t}$ is one-valued inside and on the whole contour. It is also regular inside and on the contour. By Cauchy's Theorem $\int_{ABA'C'DCA} t^{z-1}e^{-t} = 0$

i.e. $\int_{ABA'} = -\int_{CA} + \int_{C'A'} - \int_{C'DC}$

On CA $t = v$ (real and positive)

On$C'A'$ $t = ve^{2\pi i}$ (v real and positive)

Hence $\int_{CA} = \int_{\epsilon}^{R} u^{z-1}e^{-u}du$

$\int_{C'A'} = \int_{\epsilon}^{R} (ue^{2\pi i})^{z-1}e^{-u}du$

On $C'DC$ $t = \epsilon e^{i\theta}$ $0 \leq \theta \leq 2\pi$

Hence if $z = x + iy$

$t^{z-1} = e^{(x-1+iy)\log t} = e^{(x-1+iy)(\log\epsilon + i\theta)}$

$|t^{z-1}| = e^{Re(x-1+iy)(\log\epsilon+i\theta)} = e^{(x-1)\log\epsilon - y\theta)} = \epsilon^{x-1}e^{-y\theta}$

Hence $|t^{z-1}| \leq \epsilon^{x-1}e^{2\pi|y|}$

Also $|e^{-t}| = e^{-Re(t)} \leq e^{\epsilon}$

Hence $|t^{z-1}e^{-t}| \leq \epsilon^{x-1}e^{2\pi|y|+\epsilon}$

Therefore $\left|\int_{C'DC} t^{z-1}e^{-t}dt\right| \leq \epsilon^{x-1}e^{2\pi|y|+\epsilon}2\pi\epsilon = 2\pi\epsilon^{x}e^{2\pi|y|+\epsilon}$

Hence $\lim_{\epsilon\to 0}\int_{C'DC} = 0$ if $x = Re(z) > 0$

Hence when $\epsilon \to 0$

$\int_{ABA'} t^{z-1}e^{-t}dt = (e^{2\pi iz} - 1)\int_{0}^{R} u^{z-1}e^{-u}du$

Now let $R \to \infty$ so that the loops ABA' takes a limiting form as shown.

We write $\int_{\infty}^{0+}$ for this loop integral.

DIAGRAM

Since $\lim_{R\to\infty}\int_{0}^{R} u^{z-1}e^{-u}du = \Gamma(z)$ $(Re(z) > 0)$

$\Gamma(z) = \frac{1}{e^{2\pi iz} - 1}\int_{\infty}^{0+} t^{z-1}e^{-t}dt$ $(Re(z) > 0,\ z \neq$integer)

We can now dispense with the condition $Re(z) > 0$ since the path of integration does not pass through the origin. We can show by integration by parts that this new definition $z\Gamma(z) = \Gamma(z+1)$. For $Re(z) > 0$ Euler's definite integral definition is equivalent to Weierstrass's product, and also to the contour integral definition. Both the product and the contour integral satisfy $z\Gamma(z) = \Gamma(z+1)$. So both define the whole function now in the whole plane where both have a meaning. From Weierstrass's definition $\Gamma(z)\Gamma(1-z) = \pi \csc \pi z$. Hence the contour integral satisfies this equation. So

$$\begin{aligned}\frac{1}{\Gamma(1-z)} &= \Gamma(z)\frac{\sin \pi z}{\pi} = \frac{\sin \pi z}{\pi}\frac{1}{e^{2\pi i z}-1}\int_{\infty}^{0+} t^{z-1}e^{-t}dt \\ &= \frac{e^{\pi i z}-e^{-\pi i z}}{2\pi i}\frac{1}{e^{2\pi i z}-1}\int_{\infty}^{0+} t^{z-1}e^{-t}dt \\ &= \frac{1}{2\pi i e^{\pi i z}}\int_{\infty}^{0+} t^{z-1}e^{-t}dt\end{aligned}$$

Replacing z by $1-z$ we have

$$\begin{aligned}\frac{1}{\Gamma(z)} &= \frac{1}{2\pi i e^{\pi i(1-z)}}\int_{\infty}^{0+} t^{-z}e^{-t}dt \\ &= \frac{-1}{2\pi i}\int_{\infty}^{0+} (te^{-\pi i})^{-z}e^{-t}dt\end{aligned}$$

Put $s = te^{-\pi i}$ then $ds = e^{-\pi i}dt = -dt$

$\dfrac{1}{\Gamma(z)} = \dfrac{1}{2\pi i}\displaystyle\int_{-\infty}^{0+} s^{-z}e^{s}ds$

DIAGRAM

Asymptotic Behaviour of $\Gamma(z)$ as $z \to \infty$

We shall show that $\log \Gamma(z) = (z-\frac{1}{2})\log z - z + \log(2\pi)^{\frac{1}{2}} + \epsilon(z)$ where $\epsilon(z) = O(\frac{1}{z})$ in a sector $-\pi+\delta \le \arg z \le \pi - \delta$. For $z = x$ we show that $\epsilon(x) = \frac{\theta}{12x}$ $0 < \theta < 1$

i.e. $\Gamma(z) = z^{z-\frac{1}{2}}e^{-z}(2\pi)^{\frac{1}{2}}\exp\{\epsilon(z)\}$

$\Gamma(z) \sim z^{z-\frac{1}{2}}e^{-z}(2\pi)^{\frac{1}{2}}$

(Stirling's formula $n! \sim n^{n+\frac{1}{2}}e^{-n}(2\pi)^{\frac{1}{2}}$)

We have by Euler's Limit Formula.

$$\log \Gamma(z) = \lim_{n\to\infty}[z\log n + \log n! - \{\log z + \log(z+1) + \cdots + \log(z+n)\}] \quad (1)$$

Now $\displaystyle\int_r^{r+1} f(t)dt = \int_0^1 1.f(r+t)dt = \left[(t-\frac{1}{2})f(r+t)\right]_0^1 - \int_0^1 (t-\frac{1}{2}f'(r+t)dt$

$$= \frac{1}{2}\left[f(r+1)+f(r)\right] - \int_0^1 (t-\frac{1}{2})f'(r+t)dt \quad (2)$$

(N.B. This is a starting point for obtaining the Euler-Maclaurin formula for approximate integration.)

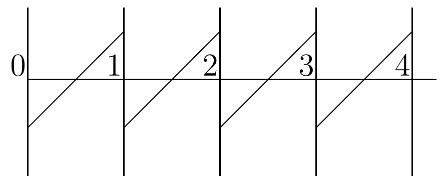
Summing from $r = o$ to $r = n-1$ we have

$$\int_0^n f(t)dt$$

$$= \frac{1}{2}f(0) + f(1) + \cdots + f(n-1) + \frac{1}{2}f(n) - \int_0^1 (t - \frac{1}{2}\sum_{r=0}^{n-1} f'(r+t)dt \quad (3)$$

Define

$$\begin{array}{ccll} \phi(t) & = & t - \frac{1}{2} & 0 \le t < 1 \\ \phi(t+1) & = & \phi(t) & t \ge 0 \end{array}$$



Then $\int_0^1 (t - \frac{1}{2}) \sum_{r=0}^{n-1} f'(r+t)dt = \int_0^1 \sum_{r=0}^{n-1} \phi(r+t)f'(r+t)dt$

$$= \int_0^n \phi(t)f'(t)dt$$

Hence we have

$$\int_0^n \log(t+z)dt$$

$$= \frac{1}{2}\log z + \log(z+1) + \cdots + \log(z+n-1) + \frac{1}{2}\log(z+n) - \int_0^n \frac{\phi(t)}{t+z}dt$$

Therefore $z\log n - \log z - \cdots - \log(z+n) + \log n!$

$$= z\log n - \left[\int_0^n \log(t+z)dt + \frac{1}{2}\log z + \frac{1}{2}\log(z+n) + \int_0^n \frac{\phi(t)}{t+z}dt + \log n!\right] \quad (4)$$

Now $\int_0^n \log(t+z)dt = [(t+z)\log(t+z) - t]_0^n = (n+z)\log(n+z) - n - z\log z$

RHS of (4)

$$= z\log n - \left(n + z + \frac{1}{2}\right) log(n+z) + \left(z - \frac{1}{2}\right)\log z + \log n! + n - \int_0^n \frac{\phi(t)}{t+z}dt$$

$$z\log n - \left(z + n + \frac{1}{2}\right)\log(n+z)$$

$$= z\log n - \left(z + n + \frac{1}{2}\right)\left[\log n + \log\left(1 + \frac{z}{n}\right)\right]$$

$$= -\left(n + \frac{1}{2}\right)\log n - \left(z + n + \frac{1}{2}\right)\log\left(1 + \frac{z}{n}\right)$$

$$= -\left(n + \frac{1}{2}\right)\log n - \left(z + n + \frac{1}{2}\right)\left[\frac{z}{n} + O\left(\frac{z^2}{n^2}\right)\right] \qquad n > |z|$$

$$= -\left(n+\frac{1}{2}\right)\log n - z + O\left(\frac{1}{n}\right)$$

where $O\left(\frac{1}{n}\right)$ involves z.

Therefore RHS of (4)

$$= \left(z-\frac{1}{2}\right)\log z - z + \left[\log n! - \left(n+\frac{1}{2}\right)\log n + n\right] - \int_0^n \frac{\phi(t)}{t+z}dt + O\left(\frac{1}{n}\right)$$

Now $\lim_{n\to\infty}$ LHS of (4)$= \log\Gamma(z)$

We show later that

$$-\epsilon(z) = \lim_{n\to\infty}\int_0^n \frac{\phi(t)}{t+z}dt = \int_0^\infty \frac{\phi(t)}{t+z}dt \qquad \text{(A)}$$

and also $\lim_{z\to\infty}\epsilon(z) = 0$ where $-\pi+\delta < \arg z < \pi-\delta$ (B)

Assuming (A) we have that $\lim_{n\to\infty}\left\{\log n! - \left(n+\frac{1}{2}\right)\log n + n\right\} = c$

(This result can be proved independently by a rather simpler method.)

Also assuming (B) we can evaluate c, we have

$$\log\Gamma(z) + \log\Gamma\left(z+\frac{1}{2}\right) + (2z-1)\log 2 - \log\Gamma(2z) = \log\Gamma\left(\frac{1}{2}\right)$$

When (A) and (B) are assumed we have

$$\log\Gamma(z) = \left(z-\frac{1}{2}\right)\log z - z + c + \epsilon(z)$$

where $\lim_{z\to\infty}\epsilon(z) = 0$

Applying this to the previous equation

$$\left[\left(z-\frac{1}{2}\right)\log z - z + c + \epsilon(z)\right] + \left[z\log\left(z+\frac{1}{2}\right) - \left(z+\frac{1}{2}\right) + c + \epsilon\left(z+\frac{1}{2}\right)\right]$$
$$+(2z-1)\log 2 - \left[\left(2z-\frac{1}{2}\right)\log 2z - 2z + c + \epsilon(2z)\right] = \log\Gamma\left(\frac{1}{2}\right)$$

Therefore

$$\log z\left[z-\frac{1}{2}+z-\left(2z-\frac{1}{2}\right)\right] + z\log\left(1+\frac{1}{2z}\right) + \left(-z-z-\frac{1}{2}+2z\right)$$
$$+c+\epsilon(z)+\epsilon\left(z+\frac{1}{2}\right)-\epsilon(2z)+(\log 2)\left[-\left(2z-\frac{1}{2}\right)+2z-1\right] = \log\Gamma\left(\frac{1}{2}\right)$$

Taking the limit as $z\to\infty$ gives

$$c+\frac{1}{2}-\frac{1}{2}-\frac{1}{2}\log 2 = \log\Gamma\left(\frac{1}{2}\right)$$

Therefore $c = \log 2^{\frac{1}{2}}\Gamma\left(\frac{1}{2}\right) = \log(2\pi)^{\frac{1}{2}}$

$$\left[\text{since } \lim_{z\to\infty} z\log\left(1+\frac{1}{2z}\right) = \lim_{z\to\infty}\log\left(1+\frac{1}{2z}\right)^z\right.$$
$$\left.= \log\lim_{z\to\infty}\left(1+\frac{1}{2z}\right)^z = \log\left(e^{\frac{1}{2}}\right) = \frac{1}{2}\right]$$

Hence we have

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + \log(2\pi)^{\frac{1}{2}} + \epsilon(z) \tag{5}$$

$$\text{where } \epsilon(z) = -\int_0^\infty \frac{\phi(t)}{t+z} dt \tag{6}$$

To prove (A)

We consider $\int_0^n \frac{\phi(t)}{t+z} dt$

Define $\phi_1(t)$ by $\phi_1'(t) = \phi(t) \quad 0 < t < 1$

$\phi_1(0) = 0 \qquad \phi_1(t+1) = \phi_1(t) \quad t \geq 0$

Therefore $\phi_1(t) = \frac{1}{2}t^2 - \frac{1}{2}t \quad (0 \leq t < 1)$

Then $\int_0^n \frac{\phi(t)}{t+z} dt = \int_0^n \frac{\phi_1'(t)}{t+z} dt = \left[\frac{\phi_1(T)}{(t+z)}\right]_0^n + \int_0^n \frac{\phi_1(t)}{(t+z)^2} dt$

$= 0 + \int_0^n \frac{\phi_1(t)}{(t+z)^2} dt$

$\phi_1(t)$ is bounded, $\qquad (0 \geq \phi_1(t) \geq -\frac{1}{8})$

and also for $t > 2|z|$ we have

$|t+z| \geq t - |z| > \frac{1}{2}t$

Therefore $\frac{1}{|t+z|^2} < \frac{4}{t^2}$

So $\int_0^n \frac{\phi_1(t)}{(t+z)^2} dt$ converges absolutely as $n \to \infty$.

This proves (A) and we have

$$\epsilon(z) = -\int_0^\infty \frac{\phi(t)}{t+z} dt = -\int_0^\infty \frac{\phi_1(t)}{(t+z)^2} dt \tag{7}$$

To prove B

I) Behaviour of $\epsilon(x)$ for real positive x.

$$\epsilon(x) = -\int_0^\infty \frac{\phi_1(t)}{(t+x)^2} dt$$

Firstly we see that $\epsilon(x) > 0$. Also $0 \leq -\phi(t) \leq \frac{1}{8}$

Therefore $0 \leq \epsilon(x) \leq \frac{1}{8} \int_0^\infty \frac{dt}{(t+x)^2} = \frac{1}{8x}$

This is sufficient to prove B for real positive x. The bound can however be improved.

Let $\phi_2(t) = \frac{1}{6}t^3 - \frac{1}{4}t^2 + \frac{1}{12}t \quad 0 \leq t < 1$

$$\phi_2(t+1) = \phi_2(t) \quad t \geq 0$$

Then $\phi_2(0) = \phi_2(1) = 0$

and $\phi_2'(t) = \phi_1(t) + \frac{1}{12}$

Therefore

$$\epsilon(x) = -\int_0^\infty \frac{-\frac{1}{12} + \phi_2'(t)}{(t+x)^2} dt = \frac{1}{12x} - \left[\frac{\phi_2(t)}{(t+x)^2}\right]_0 \infty - 2\int_0^\infty \frac{\phi_2(t)dt}{(t+x)^3} dt$$

$$= \frac{1}{12x} - 2\int_0^\infty \frac{\phi_2(t)}{(t+x)^3} dt$$

$$\int_0^\infty \frac{\phi_2(t)}{(t+x)^3} dt > 0$$

Therefore $0 \leq \epsilon(x) \leq \frac{1}{12x} \quad \epsilon(x) = \frac{\theta}{12x} \quad 0 \leq \theta \leq 1$

II Behaviour of $\epsilon(z)$ for complex z

$$\epsilon(z) = -\int_0^\infty \frac{\phi_1(t)}{(t+z)^2} dt$$

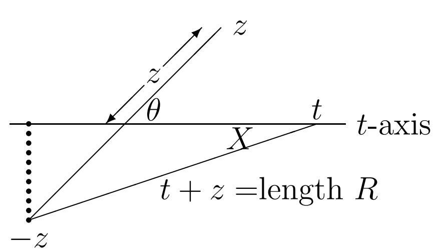
We suppose $-\pi + \delta \leq \arg z \leq \pi - \delta, \quad 0 < \delta < \pi$

$$|\epsilon(z)| \leq \int_0^\infty \left|\frac{\phi_1(t)}{(t+z)^2}\right| dt \leq \frac{1}{8}\int_0^\infty \frac{dt}{|t+z|^2}$$

$$|t+z|^2 = R^2 = (r\sin\theta)^2 \csc^2 X$$

$$t = r\sin\theta \cot X - r\cos\theta$$

$$dt = -r\sin\theta \csc^2 X dX$$



Therefore $\dfrac{dt}{|t+z|^2} = -\dfrac{dX}{r\sin\theta}$

Therefore $\displaystyle\int_0^\infty \frac{dt}{|t+z|^2} = \int_\theta^0 \frac{-dX}{r\sin\theta} = \frac{\theta}{\sin\theta}\frac{1}{r} = \frac{\theta}{\sin\theta}\frac{1}{|z|}$

The same result is obtained when $\theta < 0$

$0 \le \left| \dfrac{\theta}{\sin\theta} \right| \le \dfrac{\pi - \delta}{\sin\delta}$

Therefore $\epsilon(z)| \le \dfrac{1}{8} \dfrac{\pi - \delta}{\sin\delta} \dfrac{1}{|z|}$

Therefore $\lim\limits_{z\to\infty} \epsilon(z) = 0$ uniformly with respect to $\arg z$ for $|\arg z| \le \pi - \delta$

Some Applications of the asymptotic formula

$$\lim_{z\to\infty} \left[\log\Gamma(z) - \left(z - \frac{1}{2}\right)\log z + z - \log(2\pi)^{\frac{1}{2}} \right] = 0$$

1) $\lim\limits_{z\to\infty} \dfrac{\Gamma(z+a)}{\Gamma(z)} z^{-a} = 1$

In fact $\dfrac{\Gamma(z+a)}{\Gamma(z)} z^{-a} = 1 + O\left(\dfrac{1}{z}\right)$

$$\log \frac{\Gamma(z+a)}{\Gamma(z)} z^{-a} = -a\log z + \left(z + a - \frac{1}{2}\right)\log(z+a) - (z+a)$$
$$- \left(z - \frac{1}{2}\right)\log z + z + \epsilon(z+a) - \epsilon(z)$$
$$= \log z \left\{ -a + z + a - \frac{1}{2} - z + \frac{1}{2} \right\} + \left(z + a - \frac{1}{2}\right)\log\left(1 + \frac{a}{z}\right)$$
$$-a + \epsilon(z+a) - \epsilon(z)$$
$$= \left(z + a - \frac{1}{2}\right)\left(\frac{a}{2} + O\left(\frac{1}{z^2}\right)\right) - a + \epsilon(z+a) - \epsilon(z)$$
$$= O\left(\frac{1}{2}\right) + \epsilon(z+a) - \epsilon(z)$$

Hence the result.

2) Behaviour of $\Gamma(z)$ as $T(z) \to \pm\infty$

If $z = x + iy \quad y \to \pm\infty (x \text{ fixed})$

then $|\Gamma(x+iy)| \sim |y|^{x-\frac{1}{2}} e^{-\frac{1}{2}\pi|y|} (2\pi)^{\frac{1}{2}}$

$\Gamma(z) \to 0$ as $I(z) \to \pm\infty$ for any $R(z)$

Bernoulli Polynomials and Numbers

Define a sequence of polynomials $P_n(t)$ $n = 1, 2 \cdots$ by

$$\begin{aligned} P_1(t) &= t - \tfrac{1}{2} \\ P_2'(t) &= P_1(t) && P_2(0) = 0 \\ P_3'(t) &= P_2(t) - \bar{P}_2 && P_3(0) = 0 \\ P_4'(t) &= P_3(t) && P_4(0) = 0 \\ P_5'(t) &= P_4(t) - \bar{P}_4 && P_5(0) = 0 \end{aligned}$$

Where $\bar{P}_2 =$ mean value of $P_2(t)$ in $(0, 1)$. i.e. $\bar{P}_2 = \int_0^1 P_2(t)dt$

We show that

a) $P_1(t),\ P_3(t),\ \cdots$ are anti-symmetric about $t = \frac{1}{2}$

$P_2(t),\ P_4(t),\ \cdots$ are symmetric about $t = \frac{1}{2}$

b) $P_2(t),\ P_4(t),\ \cdots$ have zeros at $t = 0,\ t = 1$ and no others in $[0, 1]$.

$P_3(t),\ P_5(t),\ \cdots$ have zeros at $t = 0,\ t = \frac{1}{2},\ t = 1$ and no others in $[0, 1]$.

DIAGRAM

$$\begin{aligned} P_1(t) &= t - \frac{1}{2} \\ P_2(t) &= \frac{1}{2}t^2 - \frac{1}{2}t \\ P_3(t) &= \frac{1}{6}t^3 - \frac{1}{4}t^2 + \frac{1}{12}t \\ P_4(t) &= \frac{1}{24}t^4 - \frac{1}{12}t^3 + \frac{1}{24}t^2 \\ P_5(t) &= \frac{1}{120}t^5 - \frac{1}{48}t^4 + \frac{1}{72}t^3 - \frac{1}{720}t \\ \bar{P}_2 &= -\frac{1}{12} \qquad \bar{P}_4 = \frac{1}{720} \end{aligned}$$

The Bernoulli Polynomial $\phi_n(t)$ is defined by $\phi_n(t) = n!P_n(t)$.

The Bernoulli number B_n is defined by $B_n = (-1)^n(2n)!\bar{P}_{2n}$.

$B_1 = \frac{1}{6} \qquad B_2 = \frac{1}{30} \qquad B_3 = \frac{1}{42}$

Also $\dfrac{B_n}{(2n)!} = \dfrac{2S_{2n}}{(2\pi)^{2n}} \qquad S_k = \displaystyle\sum_{r=0}^{\infty} \frac{1}{r^k}$

Proof of a)

$$\frac{d}{dt}\left[P_{2k}(t) - P_{2k}(1-t)\right] = P'_{2k}(t) + P'_{2k}(1-t)$$
$$= P_{2k-1}(t) + P_{2k-1}(1-t) \qquad \text{(i)}$$

$$\frac{d}{dt}\left[P_{2k+1}(t) - P_{2k+1}(1-t)\right] = P'_{2k+1}(t) + P'_{2k+1}(1-t)$$
$$= P_{2k}(t) + P_{2k}(1-t) \qquad \text{(ii)}$$

Assume $P_{2k-1}(t)$ is anti-symmetric about $t = \frac{1}{2}$ (A)
then the RHS of (i)=0.
Therefore $P_{2k}(t) - P_{2k}(1-t) = const = P_{2k}(1) - P_{2k}(0)$
$= \int_0^1 P'_{2k}(t)dt = \int_0^1 P_{2k-1}(t) = 0$ by (A)
Therefore A$\Rightarrow$B: P_{2k} is symmetric about $t = \frac{1}{2}$.
Hence RHS of (ii)=0, and we have
$P_{2k+1}(t) + P_{2k+1}(1-t) = const = P_{2k+1}(1) + P_{2k+1}(0)$
$P_{2k+1}(0) = 0$ by the construction of $P_n(t)$
Therefore $P_{2k+1}(1) = \int_0^1 P'_{2k+1}(t)dt = \int_0^1 (P_{2k}(t) - \bar{P_{2k}})dt = 0$
Therefore $P_{2k+1}(t)$ is anti-symmetric about $t = \frac{1}{2}$.
i.e. $A(k) \Rightarrow A(k+1)$ and $B(k)$
$P_1(t) = t - \frac{1}{2}$ therefore $A(1)$ is true, hence (a) is proven by induction.

Proof of (b)

We know that $P_{2k-1}(t)$ vanishes at $t = 0$ (by construction) and at $t = \frac{1}{2}$ and $t = 1$ (by anti-symmetry about $t = \frac{1}{2}$). Also $P_{2k}(t)$ vanishes at $t = 0$ by construction and at $t = 1$ (by symmetry about $t = \frac{1}{2}$). We use the fact that if $f(t)$ is continuous and $f'(t)$ exists then $f'(t)$ has at least one zero between consecutive zeros of $f(t)$.
(A) Assume that $P_{2k-1}(t)$ has no zero other than $t = \frac{1}{2}$ in $0 < t < 1$. Now $P_{2k-1}(t) \not\equiv 0$ and is anti-symmetric about $t = \frac{1}{2}$. Hence either $P_{2k-1}(t)$ is positive in $0 < t < \frac{1}{2}$ and negative in $\frac{1}{2} < t < 1$, or vice versa.
Therefore $P'_{2k}(t) = P_{2k-1}(t)$ either steadily increases in $0 < t < \frac{1}{2}$ and steadily decreases in $\frac{1}{2} < t < 1$ or vice versa, and $P'_{2k}(\frac{1}{2}) = 0$.
$P_{2k}(0) = P_{2k}(1) = 0$, therefore $P_{2k}(t)$ has no zeros in $0 < t < 1$ (B)
Also $P_{2k}(t) - c$ has at most two zeros in $0 \leq t \leq 1$ for any c. In particular $P'_{2k+1}(t) = P_{2k}(t) - \bar{P_{2k}}$ has at most 2 zeros in $0 \leq t \leq 1$, therefore $P_{2k+1}(t)$ has no zeros in $(0, \frac{1}{2})$ or $(\frac{1}{2}, 1)$.
Therefore $A(k) \Rightarrow A(k+1)$ and $B(k)$.
A(1) is true, hence the result by induction.

N.B. The zeros of $P_3(t),\ P_5(t)\cdots$ at $0, \frac{1}{2}, 1$ are all simple.
The zeros of $P_4(t),\ P_6(t)\cdots$ at $0, 1$ are all double.
It can be proved that
$$\frac{he^{ht}-1}{e^h-1} = \sum_{n=0}^{\infty} P_n(t)h^n$$

Asymptotic Expansion of logΓ(z)

$\log\Gamma(z) = (z-\frac{1}{2})\log z - z + \log(2\pi)^{\frac{1}{2}} + \epsilon(z)$

where $\epsilon(z) = -\int_0^\infty \frac{\psi_1(t)}{t+z}dt$

Where $\psi_1(t)$ (formerly $\phi_1(t)$) is defined by

$\psi_1(t) = (t-\frac{1}{2}) \qquad (0 \le t \le 1)$

$\psi_1(t+1) = \psi_1(t) \qquad t \ge 0$

Let $\psi_n(t) = P_n(t) = \frac{\phi_n(t)}{n!} \qquad 0 \le t < 1 \qquad n = 2, 3\cdots$

$\psi_n(t+1) = \psi_n(t) \qquad t \ge 0$

Then $\psi'_{2n}(t) = \psi_{2n-1}(t)$

$\psi'_{2n+1}(t) = \psi_{2n}(t) - \bar{P_{2n}}$

Now $\int_0^\infty \frac{\psi_1(t)}{t+z}dt = \int_0^\infty \frac{\psi'_2(t)}{t+z}dt = \left[\frac{\psi_2(t)}{t+z}\right]_0^\infty + \int_0^\infty \frac{\psi_2(t)}{(t+z)^2}dt$

$\int_0^\infty \frac{\psi'_3(t)+\bar{P}_2}{(t+z)^2}dt = \frac{\bar{P}_2}{z} + \left[\frac{\psi_3(t)}{(t+z)^2}\right]_0^\infty + 2!\int_0^\infty \frac{\psi_3(t)}{(t+z)^3}dt$

$= \frac{\bar{P}_2}{z} + 2!\int_0^\infty \frac{\psi_3(t)}{(t+z)^3}dt$

Continuing this process we find

$$\int_0^\infty \frac{\psi_1(t)}{t+z}dt = \frac{\bar{P}_2}{z} + 2!\frac{\bar{P}_4}{z^3} + \cdots + \frac{(2n-2)!\bar{P_{2n}}}{z^{2n-1}} + (2n)!\int_0^\infty \frac{\psi_{2n+1}(t)}{(t+z)^{2n+1}}dt$$

Therefore $\epsilon(z) = -\sum_{r=1}^{n} \frac{(2r-2)!\bar{P_{2r}}}{z^{2r-1}} + R_n(z)$

$R_n(z) = -(2n)!\int_0^\infty \frac{\psi_{2n+1}(t)}{(t+z)^{2n+1}}dt$

$= -(2n+1)!\int_0^\infty \frac{\psi_{2n+2}(t)}{(t+z)^{2n+2}}dt$

after one further integration by parts with $\psi_{2n+1}(t) = \psi_{2n+2}(t)$.

Substituting $B_n = (-1)^n(2n)!\bar{P_{2n}}$ we get

$$\epsilon(z) = \sum_{n=1}^{m} \frac{(-1)^{n-1}B_n}{(2n-1)^{2n}}\frac{1}{z^{2n-1}} + R_m(z)$$

Magnitude of $R_n(z)$
First note that $\psi_2(t) \leq 0, \ \ \psi_2(t) \leq 0$ etc.
Let $z = x$, real and positive. In this case $R_0(x) > 0, \ \ R_1(x) < 0$ etc. i.e. the $R_n(x)$ are alternately positive and negative. Hence $\epsilon(x)$ lies between the sums to n and $n+1$ terms of the series
$\frac{B_1}{1.2}\frac{1}{x} - \frac{B_2}{3.4}\frac{1}{x^3} + \cdots$
and $R_n(x)$ is numerically less than the term $\frac{(-1)^n B_{n+1}}{(2n+1)(2n+2)}\frac{1}{x^{2n+1}}$
In particular $\lim_{x\to\infty} x^{2n-1}R_n(x) = 0$ (n fixed) (1)
In fact $\lim_{x\to\infty} x^{2n+1}R_n(x)$ exists and $= \frac{(-1)^n B_{n+1}}{(2n+1)(2n+2)}$ (2)
The property (1) characterises the asymptotic nature of the series $\frac{B_1}{1.2}\frac{1}{x} - \cdots$

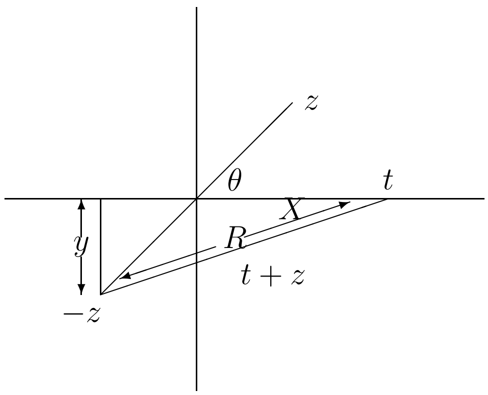
Divergence of the series
The above series taken to ∞ is divergent for all x. i.e. $\lim_{n\to\infty} R_n(x)$ does not exist.
This follows the result (here quoted but not proved) that
$(-1)^n \bar{P}_{2n} = \frac{B_n}{(2n)!} = \frac{2S_{2n}}{(2\pi)^{2n}}$
$\frac{B_n}{(2n)!} \sim \frac{2}{(2\pi)^{2n}}$
In that case the nth term of the series is
$(-1)^n(2n-2)!\frac{2S_{2n}}{(2\pi)^{2n}}\frac{1}{z^{2n-1}} \sim \frac{2(-1)^n}{2\pi}\frac{(2n-2)!}{(2\pi z)^{2n-1}}$
And $\sum_{n=1}^{\infty} \frac{(-1)^n(2n-2)!}{z^{2n-1}}$ diverges for all z.
Hence the result.
We write $\epsilon(z) = \frac{B_1}{1.2}\frac{1}{z} - \frac{B_2}{3.4}\frac{1}{z^3} + \cdots$ and
$\log\Gamma(z) \sim \left(z - \frac{1}{2}\right)\log z - z + \log(2\pi)\frac{1}{2} + \frac{B_1}{1.2}\frac{1}{z} - \frac{B_2}{3.4}\frac{1}{z^3} + \cdots$

Behaviour of $R_n(z) - z$ complex -as $z \to \infty$
We have $R_n(z) = -(2n+1)!\int_0^\infty \frac{\psi_{2n+2}(t)}{(t+z)^{2n+2}}dt$
We suppose as before that $|\arg z| \leq \pi - \delta, \ \ (0 < \delta < \pi)$ necessary for the existence of $R_n(z)$.

a) when $x = Re(z) > 0$

$|t+z| \geq t+x$ for all $t \geq 0$ and hence

$$|R_n(z)| \leq (2n+1)! \int_0^\infty \frac{|\psi_{2n+2}(t)|}{(t+x)^{2n+2}} dt$$



Either $\psi_{2n+2}(t) \leq 0$ $\quad n = 0, 2, 4, \cdots$

or $\quad \psi_{2n+2}(t) \geq 0$ $\quad n = 1, 3, 5, \cdots$

$|R_n(z)| \leq |R_n(x)|$

If m_n is the upper bound of $|\psi_{2n}(t)|$

$$|R_n(z)| \leq \frac{(2n+1)! m_{n+2}}{(2n+1)x^{2n+1}} = \frac{K_n}{x^{2n+1}} = \frac{K_n \sec^{2n+1}\theta}{|z|^{2n+1}}$$

Hence $|R_n(z)| = O(|z|^{-(2n+1)})$ $\quad |\arg z| < \frac{\pi}{2}$

b) when $|\arg z| \leq \pi - \delta$

$$|R_n(z)| \leq (2n+1)! \int_0^\infty \frac{|\psi_{2n+2}(t)|}{|t+z|^{2n+2}} dt$$

$|t+z| = y \csc X$ $\quad t = y \cot X - x$

$$|R_n(z)| \leq (2n+1)! \int_0^\pi \frac{|\psi_{2n+2}(t)| y \csc^2 X dX}{y^{2n+2} \csc^{2n+2} X}$$

$$\leq \frac{(2n+1)! m_{n+2}}{y^{2n+1}} \int_0^\pi \sin^{2n} X dX$$

$$= \frac{K'_n}{y^{2n+1}} = \frac{K'_n \csc^{2n+1}\theta}{|z|^{2n+1}}$$

$$\leq \frac{K_n' \csc^{2n+1} \delta}{|z|^{2n+1}} \qquad |\theta| \leq \pi - \delta$$

$$|R_n(z)| = O(z^{-(2n+1)})$$