Fourier Series and their Applications

The functions $1, \cos x, \sin x, \cos 2x, \sin 2x, \cdots$ are orthogonal over $(-\pi, \pi)$.

$$\int_{-\pi}^{\pi} \cos mx \cos nx dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \neq 0 \\ 2\pi & m = n = 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx dx = 0 \text{ for all } m, n$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \neq 0 \end{cases}$$

In fact the functions satisfy these relations over any interval $(\alpha, \alpha + 2\pi)$. Assuming that $f(x)$, defined and integrable in $(-\pi, \pi)$, has an expansion.

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos mx + b_n \sin nx)$$

uniformly convergent over $(-\pi, \pi)$

$$\int_{-\pi}^{\pi} f(x)dx = \pi a_0$$

$$\int_{-\pi}^{\pi} f(x) \cos nx dx = \pi a_n \text{ therefore } a_n + ib_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx$$

$$\int_{-\pi}^{\pi} f(x) \sin nx dx = \pi b_n$$

These coefficients exist irrespective of whether or not the series converges and is equal to $f(x)$, and they are called the Fourier coefficients.

Sufficient Conditions for convergence

a) If $f(x)$ is differentiable at ξ (or if $\exists m$, such that $\left|\dfrac{f(x) - f(\xi)}{x - \xi}\right| < m$, $x \epsilon (\xi - h, \xi + h)$) then the fourier series converges at ξ to $f(\xi)$.

b) If $f(x)$ is monotonic in $\xi < x < \xi + h$ and in $\xi - h < x < \xi$ for some $h > 0$, then the fourier series converges at ξ to the value $\frac{1}{2}\{f(\xi - 0) + f(\xi + 0)\}$.

General Range

The range $a \le x \le b$ is standardised by substituting $X = \dfrac{\pi\left(x - \frac{a+b}{2}\right)}{\left(\frac{b-a}{2}\right)}$

then $-\pi < X < \pi$.

The series $\dfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nX + b_n \sin nX$

becomes $\dfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\pi \left(\dfrac{2x-(a+b)}{b-a}\right) + b_n \sin 2n\pi \left(\dfrac{2x-(a+b)}{b-a}\right)$

Periodicity of $f(x)$

We suppose that $f(x)$ is represented by the series (when cgt.) for all x, hence since the sum function of the series is periodic, with period 2π, we have $f(x+2\pi) = f(x)$ which defines $f(x)$ outside the original range.

Fourier Series for $\frac{1}{2} - t \qquad (0 < t < 1)$

First consider the identity

$$1 + \sum_{n=1}^{m} e^{nix} = \frac{e^{(m+1)ix} - 1}{e^{ix} - 1} = \frac{e^{\left(m+\frac{1}{2}\right)ix} - e^{-\frac{1}{2}ix}}{2i \sin \frac{1}{2}x}$$

$$= \frac{\cos\left(m+\frac{1}{2}\right)x + i\sin\left(m+\frac{1}{2}\right)x - \left(\cos\frac{1}{2}x - i\sin\frac{1}{2}x\right)}{2i\sin\frac{1}{2}x}$$

Hence for x real $(x \neq 0, \pm 2\pi, \cdots)$ taking real and imaginary parts:

$$Re: \quad 1 + \sum_{n=1}^{m} \cos nx = \frac{1}{2} + \frac{1}{2}\frac{\sin\left(m+\frac{1}{2}\right)x}{\sin\frac{1}{2}x}$$

$$\text{or } \frac{1}{2} + \sum_{n=1}^{m} \cos nx = \frac{1}{2}\frac{\sin\left(m+\frac{1}{2}\right)x}{\sin\frac{1}{2}x} \tag{1}$$

$$Im: \quad -\frac{1}{2}\cot\frac{1}{2}x + \sum_{n=1}^{m}\sin nx = -\frac{1}{2}\frac{\cos\left(m+\frac{1}{2}\right)x}{\sin\frac{1}{2}x} \tag{2}$$

Integrate (1) and (2) from x to π

$$(1):\quad \frac{1}{2}(\pi - x) - \sum_{n=1}^{m}\frac{\sin nx}{n} = \frac{1}{2}\int_x^{\pi}\frac{\sin\left(m+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}dt \tag{3}$$

$$(2):\quad \left[-\log\left(\sin\frac{1}{2}t\right)\right]_x^{\pi} + \left[\sum_{n=1}^{m}\frac{-\cos nt}{n}\right]_x^{\pi} = -\frac{1}{2}\int_x^{\pi}\frac{\cos\left(m+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}dt \tag{4}$$

Now suppose $\delta \le x \le 2\pi - \delta$.

Then using the Riemann Lebesgue theorem, we have, letting $m \to \infty$ in (3) and (4)

$$\frac{1}{2}(\pi - x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n} \tag{5}$$

$$\log \sin \frac{1}{2}x - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} + \sum_{n=1}^{\infty} \frac{\cos nx}{n} = 0$$

Therefore $\log 2 \sin \frac{1}{2}x = -\sum_{n=1}^{\infty} \frac{\cos nx}{n}$ (6)

Alternative Proof of (5) and (6)

$$\log(1-\xi) = -\int_0^{\xi} \frac{dt}{1-t}$$

where we take a cut along the positive real axis in the t-plane from 1 to ∞. The branch of $\log(1-\xi)$ chosen is that which is real when ξ is real, and is one-valued in the cut plane. In particular this vanishes at $\xi = 0$

$$\frac{1}{1-t} = 1 + t + \cdots + t^{m-1} + \frac{t^m}{1-t}$$

therefore $\int_0^{\xi} \frac{dt}{1-t} = \sum_{n=1}^{m} \frac{\xi^n}{n} + \int_0^{\xi} \frac{t^m}{1-t} dt$

where the path is taken along the radius $0 - \xi$.

DIAGRAM

For all t on the radius through ξ

$$\begin{aligned} |1-t| &\geq |\sin\theta| \quad Re(\xi) > 0 \\ &\geq 1 \quad \text{otherwise} \end{aligned}$$

Hence in all cases $|1-t| \geq \sin\delta$ when $\delta \leq \arg\xi \leq 2\pi - \delta \qquad (0 < \delta < \pi)$

Therefore $\left| \int_0^{\xi} \frac{t^m}{1-t} dt \right| = \left| \int_0^{r} \frac{(pe^{i\theta})^m e^{i\theta}}{1-t} dp \right|$

$$\leq \int_0^r \frac{p^m}{|\sin\delta|} dp = \frac{1}{\sin\delta} \frac{r^{m+1}}{m+1} \leq \frac{1}{(m+1)\sin\delta} \qquad 0 \leq r \leq 1$$

Hence $\lim_{m\to\infty} \left| \int_0^{\xi} \frac{t^m dt}{1-t} \right| = 0 \qquad \delta \leq \arg\xi \leq 2\pi - \delta, \quad 0 \leq r \leq 1$

When $r = 1$ the convergence is uniform with respect to θ. Hence we have

$$\log(1-\xi) = -\sum_{n=1}^{\infty} \frac{\xi^n}{n}$$

Where the series converges on $|\xi| = 1$ except at $\xi = 1$, uniformly in $\delta \leq \arg\xi \leq 2\pi - \delta$

$\log(1-\xi) = \log|1-\xi| + i\arg(1-\xi)$
$= \log\left(2\sin\frac{1}{2}\theta\right) - i\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$
Taking real and imaginary parts gives
$\frac{1}{2}\pi - \theta = \sum_1^\infty \frac{\sin n\theta}{n}$
$\log\left(2\sin\frac{1}{2}\theta\right) = -\sum_1^\infty \frac{\cos n\theta}{n} \qquad 0 < \theta < 2\pi$
Convergence being uniform in $\delta \le \theta \le 2\pi - \delta$.

Fourier Expansion of the Bernoulli polynomials in $0 \le t \le 1$. Values of the Bernoulli numbers.
Put $x = 2\pi t$ in $\frac{1}{2}(\pi - x) = \sum_1^\infty \frac{\sin nx}{n}$
Therefore $t - \frac{1}{2} = -\frac{1}{\pi}\sum_1^\infty \frac{\sin 2\pi nt}{n}$

$$= -2\sum_1^\infty \frac{\sin 2\pi nt}{2\pi n} \qquad (1) \qquad 0 < t < 1$$

Therefore $P_1(t) = -2\sum_1^\infty \frac{\sin 2\pi nt}{2\pi n}$
$P_2'(t) = P_1(t) \qquad P_2(0) = 0$
Therefore $P_2(t) = \int_0^t P_1(s)ds$
The series (1) converges uniformly in $\epsilon \le t \le 1-\epsilon$

$$\int_\epsilon^t P_1(s)ds = -2\sum_1^\infty \int_\epsilon^t \frac{\sin 2n\pi s}{2n\pi}ds \qquad \epsilon \le t \le 1-\epsilon$$

$$= -2\sum_1^\infty \frac{\cos 2n\pi\epsilon - \cos 2n\pi t}{(2n\pi)^2}$$

The series on the right converges absolutely and uniformly since
$\left|\frac{\cos(2n\pi t)}{(2n\pi)^2}\right| \le \frac{1}{(2n\pi)^2}$ and $\sum \frac{1}{n^2}$ converges.
Hence $\int_0^t P_1(s)ds = -2\sum_1^\infty \frac{1-\cos 2n\pi t}{(2n\pi)^2} \qquad 0 \le t \le 1$
(using continuity)

$$\text{Hence } P_2(t) = \bar{P}_2 + 2\sum_1^\infty \frac{\cos 2n\pi t}{(2n\pi)^2} \qquad (2)$$

$\bar{P_2} = -2\sum_1^\infty \frac{1}{(2n\pi)^2} = \frac{-2}{(2\pi)^2} S_2$

Next $P_3'(t) = P_2(t) - \bar{P_2}$

Therefore $P_3(t) = 2\sum_1^\infty \frac{\sin 2n\pi t}{(2n\pi)^3}$

and generally we have

$P_{2m}(t) - \bar{P_{2m}} = (-1)^{m-1} 2 \sum_{m=1}^\infty \frac{\cos 2n\pi t}{(2n\pi)^{2m}}$

$P_{2m+1}(t) = (-1)^{m-1} 2 \sum_{m=1}^\infty \frac{\sin 2n\pi t}{(2n\pi)^{2m+1}}$

$\bar{P_{2m}} = (-1)^m \frac{2S_{2m}}{(2\pi)^{2m}}$

We also have

$\frac{\phi_m(t)}{m!} = P_m(t) \qquad m = 2, 3, \cdots$

$\frac{B_m}{(2m)!} = (-1)^m \bar{P_{2m}}$

For $k \geq 2$ it can be shown that

$1 \leq S_k \leq 1 + \frac{1}{2^{k-2}}(S_2 - 1)$

Therefore $S_k = 1 + o(k)$

Also $P_{2m+1}(t) \sim (-1)^{m-1} \frac{2}{(2\pi)^{2m+1}} \sin 2\pi t$

$P_{2m}(t) \sim (-1)^{m-1} \frac{2}{(2\pi)^m} (1 - \cos 2\pi t)$

Fourier Series of the Square Wave

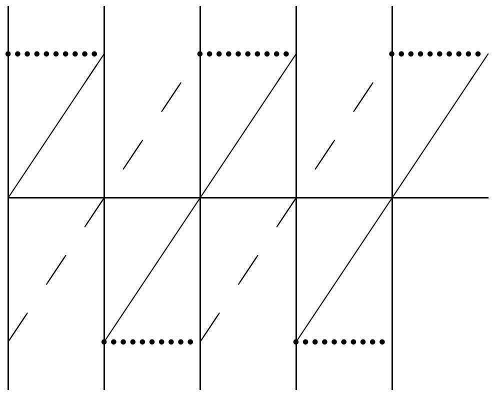
We have $\frac{1}{2}(\pi - x) = \sum_{n=1}^\infty \frac{\sin nx}{n} \qquad 0 < x < 2\pi$

Write $y = x - \pi$

$-\frac{1}{2}y = \sum_{n=1}^\infty (-1)^n \frac{\sin ny}{n} \qquad -\pi < y < \pi$

$x = 2\sum_{n=1}^\infty (-1)^{n-1} \frac{\sin nx}{n} \qquad -\pi < x < \pi$

Write $f(x) = 2\sum_{n=1}^\infty (-1)^{n-1} \frac{\sin nx}{n}$



Graph of $f(x)$ is shown by solid lines.
Graph of $f(x+\pi)$ is shown by broken lines.
Graph of $f(x) - f(x+\pi)$ is shown by dotted lines.
The fourier series of $f(x) - f(x+\pi)$ is then

$$2\sum_{n=1}^{\infty}(-1)^n \frac{\sin nx}{n} - 2\sum_{n=1}^{\infty}(-1)^{n-1}(-1)^n \frac{\sin nx}{n}$$

$$= 4\sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1}$$

$$\frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1} = \begin{cases} +1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases}$$

Find coefficients by direct integration.

Gibbs' Phenomenon

Write $S_m(x) = \dfrac{4}{\pi}\sum_{n=0}^{m} \dfrac{\sin(2n+1)x}{2n+1}$

$$\frac{\pi}{4}\frac{d}{dx}S_m(x) = \sum_{n=0}^{m} \cos(2n+1)x = \frac{\sin(2m+2)x}{2\sin x}$$

This vanishes in $o < x < \pi$ at $x = \dfrac{\pi}{2m+2} \cdots \dfrac{(2m+1)\pi}{2m+2}$

$S_m(x)$ is symmetrical about $\frac{\pi}{2}$.
Hence consider the value of S_m for $0 < x < \frac{\pi}{2}$, and in particular at $x = \dfrac{\pi}{2m+2}$, the first max.

$$\frac{\pi}{4}S_m\left(\frac{\pi}{2m+2}\right) = \int_0^{\frac{\pi}{2m+2}} \frac{\sin(2m+2)t}{2\sin t}dt$$

Put $t = \dfrac{s}{2m+2}$ then we have

$$\frac{\pi}{4}\sin\left(\frac{\pi}{2m+2}\right)=\frac{1}{2}\int_0^{\pi}\frac{\sin s ds}{(2m+2)\sin\frac{s}{2m+2}}=\frac{1}{2}\int_0^{\pi}\frac{sins}{s}\phi\left(\frac{s}{2m+2}\right)ds$$

where $\phi(u)=\dfrac{u}{\sin u}$.

Now $1\le\phi(u)\le\phi(\delta)\quad 0\le u\le\delta<\pi\qquad$ and $0\le\dfrac{s}{2m+2}\le\pi 2m+2$

So $1\le\phi\left(\dfrac{s}{2m+2}\right)\le\phi\left(\dfrac{\pi}{2m+2}\right)$

$1\ge\dfrac{\sin s}{s}\ge 0\qquad$ in $0\le s\le\pi$

Hence $\displaystyle\int_0^{\pi}\frac{\sin s}{s}ds\le\int_0^{\pi}\frac{\sin s}{s}\phi\left(\frac{s}{2m+2}\right)ds$

$\displaystyle\le\phi\left(\frac{\pi}{2m+2}\right)\int_0^{\pi}\frac{\sin s}{s}ds$

Since $\displaystyle\lim_{m\to\infty}\phi\left(\frac{\pi}{2m+2}\right)=1$ we have

$$\lim_{m\to\infty}\int_0^{\pi}\frac{\sin s}{s}\phi\left(\frac{s}{2m+2}\right)ds=\int_0^{\pi}\frac{\sin s}{s}ds$$

Hence $\displaystyle\lim_{m\to\infty}S_m\left(\frac{\pi}{2m+2}\right)=\frac{2}{\pi}\int_0^{\pi}\frac{\sin s}{s}ds\approx 1.179>1$

Dirichlet's Formula (sufficient conditions for convergence)
Assume that $f(x)$ is bounded and integrable over $[-\pi,\pi]$, and
$f(x+2\pi)=f(x)$

Write $\displaystyle S_m(x)=\frac{1}{2}a_0+\sum_{n=1}^{m}(a_n\cos nx+b_n\sin nx)$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\left[\frac{1}{2}+\sum_{n=1}^{m}(\cos nt\cos nx+\sin nt\sin nx)\right]dt$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\left[\frac{1}{2}+\sum_{n=1}^{m}\cos n(t-x)\right]dt$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\frac{\sin\left(m+\frac{1}{2}\right)(t-x)}{\sin\frac{1}{2}(t-x)}dt$$

$$=\frac{1}{2\pi}\int_{-\pi-x}^{\pi-x}f(x+s)\frac{\sin\left(m+\frac{1}{2}\right)s}{\sin\frac{1}{2}s}ds$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x+s)\frac{\sin\left(m+\frac{1}{2}\right)s}{\sin\frac{1}{2}s}ds\qquad\text{by periodicity}$$

$$=\frac{1}{2\pi}\int_0^{\pi}[f(x+t)+f(x-t)]\frac{\sin\left(m+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}dt\qquad\text{as }\frac{\sin\left(m+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\text{ is even.}$$

Since $\dfrac{1}{2}+\displaystyle\sum_{n=1}^{m}\cos nx=\dfrac{\sin\left(m+\frac{1}{2}\right)x}{2\sin\frac{1}{2}x}$,

$\dfrac{1}{2}\pi=\displaystyle\int_0^\pi \dfrac{\sin\left(m+\frac{1}{2}\right)x}{2\sin\frac{1}{2}x}dx$

Therefore $\dfrac{1}{2}[f(x+0)+f(x-0)]=\dfrac{1}{2\pi}\displaystyle\int_0^\pi [f(x+0)+f(x-0)]\dfrac{\sin\left(m+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}$

Therefore $S_m(x)-\dfrac{1}{2}[f(x+0)+f(x-0)]$

$$=\frac{1}{2\pi}\int_0^\pi [f(x+t)-f(x+0)]\frac{\sin\left(m+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}dt$$
$$+\frac{1}{2\pi}\int_0^\pi [f(x-t)-f(x-0)]\frac{\sin\left(m+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}dt \qquad (1)$$

When $f(x+0)=f(x-0)=f(x)$ (1) becomes

$$S_m(x)-f(x)=\frac{1}{2\pi}\int_0^\pi [f(x+t)+f(x-t)-2f(x)]\frac{\sin\left(m+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}dt \qquad (1a)$$

The integrals appearing in (1) and (1a) are all of the form

$\displaystyle\int_a^b \phi(t)\sin\lambda t dt$ where $a=0,\ b=\pi,\ \lambda=m+\frac{1}{2}$

$\phi(t)=\dfrac{f(x+t)-f(x+0)}{\sin\frac{1}{2}t},\ \ \dfrac{f(x-t)-f(x-0)}{\sin\frac{1}{2}t}$, or

$\dfrac{f(x+t)+f(x-t)-2f(x)}{\sin\frac{1}{2}t}$

Hence if $\phi(t)$ is bounded and integrable over $[0,\pi]$, then by the Riemann Lebesgue theorem, $\displaystyle\int_0^\pi \phi(t)\sin\lambda t\to 0$ as $\lambda\to\infty$.

In fact in the above cases $\phi(t)$ is bounded and integrable over $[h,\pi]$ $h>0$ and so the convergence depends only on the behaviour of the function in a sufficiently small interval $[0,h]$.

Integration of a Fourier Series

If $f(x)$ is bounded and integrable in $[-\pi,\pi]$ and $F(x)=\displaystyle\int_{-\pi}^{\pi}\left(f(t)-\frac{1}{2}a_0\right)dt$

Where $\frac{1}{2}a_0=\bar{f}$ is the constant term in the Fourier series for f, then $F(x)$ has a Fourier series, convergent everywhere to $F(x)$, obtained by integrating the Fourier series for $f(x)-\frac{1}{2}a_0$ term by term. [This holds even if the Fourier series for f does not converge.]

$F(x)$ is an absolutely continuous function and hence possesses a Fourier series converging everywhere to $F(x)$.

$F(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n \cos nx + B_n \sin nx)$

Assuming that f is continuous on $(-\pi, \pi)$ ensures the existence of $F'(x)$, and

$$
\begin{aligned}
A_n &= \frac{1}{\pi}\int_{\pi}^{\pi} F(x)\cos nx dx \qquad n = 1, 2, \cdots \\
&= \frac{1}{\pi}\left[F(x)\frac{\sin nx}{n}\right]_{-\pi}^{\pi} - \frac{1}{n}\frac{1}{\pi}\int_{-\pi}^{\pi} F'(x)\sin nx dx \\
&= 0 - \frac{1}{n\pi}\int_{-\pi}^{\pi}\left(f(x) - \frac{1}{2}a_0\right)\sin nx dx \qquad [F(\pi) = F(-\pi) = 0] \\
&= -\frac{1}{n\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = -\frac{b_n}{n}
\end{aligned}
$$

Similarly $B_n = \frac{a_n}{n}$

Therefore $F(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} -\frac{b_n}{n}\cos nx + \frac{a_n}{n}\sin nx$

Putting $x = \pi$ gives $\frac{1}{2}(a_0) = \sum_{n=1}^{\infty}\frac{b_n}{n}(-1)^n$

$$
\begin{aligned}
\text{Therefore } F(x) &= \sum_{n=1}^{\infty}\frac{a_n \sin nx + b_n((-1)^n - \cos nx)}{n} \\
&= \sum_{n=1}^{\infty}\int_{-\pi}^{x}\{a_n \cos nt + b_n \sin nt\}dt
\end{aligned}
$$

Differentiation of a Fourier Series

This is not always valid.

e.g. $\sum_{n=1}^{\infty}\frac{\sin nx}{n} = \frac{1}{2}(\pi - x) \qquad 0 \le x \le 2\pi$

$\sum_{n=1}^{\infty}\frac{d}{dx}\frac{\sin nx}{n} = \sum_{n=1}^{\infty}\cos nx \qquad$ which does not converge.

Sufficient Conditions

If $f(x)$ is continuous and $f'(x)$ exists except at a finite number of points, and both $f(x)$ and $f'(x)$ have Fourier series which converge, then the series for $f'(x)$ is obtained by term by term differentiation of the Fourier series for $f(x)$.

i.e. $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

$$\Rightarrow \frac{1}{2}[f'(x+0)+f'(x-0)] = \sum_{n=1}^{\infty} nb_n \cos nx - na_n \sin nx$$

[This is really just the same as the result for integration, with slightly weaker conditions.]

Half-Range Series

Let $f(x)$ be bounded and integrable in $[0, \pi]$

(1) Cosine Series

define $f_c(x) = \begin{cases} f(x) & 0 \le x \le \pi \\ f(-x) & -\pi \le x \le 0 \end{cases}$

Then $f_c(x)$ is an even function, which has a Fourier series in which $b_n \equiv 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_c(x)\cos nx dx$$
$$= \frac{2}{\pi}\int_0^{\pi} f_c(x)\cos nx dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$$

(2) Sine Series

define $f_s(x) = \begin{cases} f(x) & 0 < x < \pi \\ -f(-x) & -\pi < x < 0 \end{cases}$

If $f(0) \neq 0$ f_s is discontinuous at 0.

If $f(\pi) \neq 0$ f_s is discontinuous at π.

Then $f_s(x)$ is an odd function, and has a Fourier series in which $a_n \equiv 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_s(x)\sin nx dx$$
$$= \frac{2}{\pi}\int_0^{\pi} f_s(x)\sin nx dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$$

Order of magnitude of Fourier coefficients

$$a_n - ib_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = c_n \qquad (1)$$

Suppose $f(x)$ and all its derivatives are bounded and continuous in $(-\pi, \alpha_1), (\alpha_1, \alpha_2) \cdots (\alpha_k, \pi)$

Write $c_n^m = \frac{1}{\pi}\int_{-\pi}^{\pi} f^{(m)}(x)e^{-inx}dx$

$$= \frac{1}{\pi}\sum_{0}^{k}\int_{-\alpha_r}^{\alpha_{r+1}} f^{(m)}(x)e^{-inx}dx \qquad (2)$$

Integrating (1) by parts gives

$$\pi c_n = \pi c_n^0 = \sum_{r=0}^{k}\left[-\frac{f(x)}{in}e^{-inx}\right]_{\alpha_r}^{\alpha_{r+1}} + \frac{1}{in}\int_{\alpha_r}^{\alpha_{r+1}} f'(x)e^{inx}dx$$

$$= \frac{1}{in}\left[\sum_{r=0}^{k} f(\alpha_r+0)e^{-in\alpha r} - f(\alpha_{r+1}-0)e^{-in\alpha_{r+1}} + \int_{\alpha_r}^{\alpha_{r+1}} f'(x)e^{-inx}dx\right]$$

$$= \frac{1}{in}\left[f(-\pi+0)e^{-in\pi} - f(\pi-0)e^{-in\pi} + \sum_{r=1}^{k}\{f(\alpha_r+0)-f(\alpha_r-0)\}e^{-in\alpha_r} + \pi c_n^{(1)}\right]$$

$f(-\pi+0) = f(\pi+0)$ by periodicity

Therefore $f(-\pi+0)e^{in\pi} - f(\pi-0)e^{-in\pi}$

$$= [f(\pi+0)-f(\pi-0)]e^{-in\pi} = [f(\alpha_{k+1}+0)-f(\alpha_{k+1}-0)]e^{in\alpha_{k+1}}$$

Hence we have

$$\pi c_n^{(0)} = \frac{\pi c_n^{(1)}}{ni} + \frac{1}{ni}\sum_{r=1}^{k+1}\{f(\alpha_r+0)-f(\alpha_r-0)\}e^{-in\alpha_r}$$

Write $J_n^{(m)} = \frac{1}{\pi}\sum_{r=1}^{k+1}\{f^{(m)}(\alpha_r+0)-f^{(m)}(\alpha_r-0)\}e^{-in\alpha_r}$

Therefore $c_n^{(0)} = \frac{c_n^{(1)}}{ni} + \frac{J_n^{(0)}}{ni}$

Similarly $c_n^{(1)} = \frac{c_n^{(2)}}{ni} + \frac{J_n^{(1)}}{ni}$

Therefore $c_n^{(0)} = \frac{J_n^{(0)}}{ni} + \frac{J_n^{(1)}}{(ni)^2} + \frac{c_n^{(2)}}{(ni)^2}$

Since $c_n^{(2)} = \frac{1}{\pi}\int_{-\pi}^{\pi} f''(x)e^{-inx}dx$ is bounded,

if $J_n^{(0)} = 0$ for all $n \geq m$ then $c_n = c_n^{(0)}$ is $O\left(\frac{1}{n^2}\right)$ as $n \to \infty$

If also $J_n^{(1)} = 0$ for all $n \geq m$ then $c_n = c_n^{(0)}$ is $O\left(\frac{1}{n^3}\right)$ as $n \to \infty$

In particular if $f, f^{(1)}, \cdots f^{(r)}$ are continuous but $f^{(r+1)}$ is not continuous then $c_n = O\left(\frac{1}{n^{r+2}}\right)$ as $n \to \infty$

In fact $J_n^{(0)}$ vanishes only if f is continuous for if we write

$f(\alpha_r+0) - f(\alpha_r-0) = j_r$ then $\pi J_n^{(0)} = \sum_{r=1}^{k+1} j_r e^{-in\alpha_r}$.

If $J_n^{(0)} = 0$ for $n \geq m$ then

$$\sum_{r=1}^{k+1} j_r e^{-in\alpha_r} = 0 \qquad n = m, m+1, \cdots$$

Taking $n = m, m+1, \cdots, m+k$, we write $e^{-i\alpha_r} = z_r$

Therefore $\begin{pmatrix} z_1^m & z_2^m & \cdots & z_{k+1}^m \\ \vdots & & & \\ z_1^{m+k} & z_2^{m+k} & \cdots & z_{k+1}^{m+k} \end{pmatrix} \begin{pmatrix} j_1 \\ \vdots \\ j_{k+1} \end{pmatrix} = 0$

The determinant of the matrix is

$$(z_1 z_2 \cdots z_{k+1})^m \begin{vmatrix} 1 & \cdots & 1 \\ z_1 & \cdots & z_{k+1} \\ \vdots & & \\ z_1^k & \cdots & z_{k+1}^k \end{vmatrix} = (z_1 z_2 \cdots z_{k+1})^m \prod_{r>s} (z_r - z_s)$$

$z_r - z_s \neq 0$ for $r \neq s$

Therefore the determinant is non zero.

Therefore $j_1 = j_2 = \cdots = j_{k+1} = 0$

Therefore f is continuous.

Parseval's Theorem

If $f(x)$ is bounded and integrable in $(-\pi, \pi)$

then $\frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2 dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$.

[Note that this is true even though $f(x)$ does not equal the sum of its Fourier series.]

If we assume that $f(x)$ is continuous and the Fourier series converges to $f(x)$,

$$\frac{1}{n}\int_{-\pi}^{\pi} f^2(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\left[\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right] dx$$

Since uniformity of convergence is not affected by multiplying by $f(x)$ we can integrate term by term

$$\text{RHS} = \frac{1}{2}a_0 \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx + \frac{1}{\pi}\sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} f(x)\cos nx dx + b_n \int_{-\pi}^{\pi} f(x)\sin nx dx$$

$$= \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2$$