Asymptotic Expansions

If $f(z) = a_0 + \frac{a_1}{z} + \cdots + \frac{a_n}{z^n} + R_n(z)$
where $\lim_{z\to\infty} z^n R_n(z) = 0$, (and $\arg z$ is suitably restricted), then the series is said to be the asymptotic expansion of $f(z)$ (valid in the appropriate sector of the z-plane).
We write $f(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$
$a_0 = \lim_{z\to\infty} f(z) \qquad a_1 = \lim_{z\to\infty}(f(z) - a_0)z \qquad \cdots$
$a_n = \lim_{z\to\infty} \left(f(z) - a_0 - \frac{a_1}{z} - \cdots - \frac{a_{n-1}}{z^{n-1}}\right)$
This is not necessarily the simplest way of finding the a_n in special cases. In many cases $a_0 + \frac{a_1}{z} + \cdots$ taken to ∞ diverges.

Properties

i) If $f(z)$ possesses such an expansion it is unique (since the a_n are uniquely determined).

ii) Non-uniqueness of a function with a given expansion.

Consider $e^{-z} \qquad |\arg z| \leq \frac{\pi}{2} - \delta$

$\lim_{z\to\infty} z^n e^{-z} = 0 \qquad$ for all n

Hence e^{-z} has no asymptotic expansion in the sense that the coefficients are all zero.

Hence if $f(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$

Then $f(z) + e^{-z} \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$

iii) Multiplication

If $f(z) \sim \sum_{n=0}^{\infty} \frac{a_n}{z^n} \qquad$ and $g(z) \sim \sum_{n=0}^{\infty} \frac{b_n}{z^n}$

Then $f(z)g(z) \sim \sum_{n=0}^{\infty} \frac{c_n}{z^n} \qquad c_n = a_n b_0 + a_{n-1} b_1 + \cdots + a_0 b_n$

Write $f(z) = a_0 + \frac{a_1}{z} + \cdots + \frac{a_n}{z^n} + R_n(z)$

$$g(z) = b_0 + \frac{b_1}{z} + \cdots + \frac{b_n}{z^n} + S_n(z)$$

(Multiplying and collecting terms gives the result.)

iv) Integration

If $f(z) \sim \sum_{n=0}^{\infty} \frac{a_n}{z^n}$

we consider $F(z) = \left[f(z) - a_0 - \frac{a_1}{z}\right] \sim \sum_{n=2}^{\infty} \frac{a_n}{z^n}$

$$\int_z^{\infty} F(w)dw \sim \sum_{n=2}^{\infty} \int_z^{\infty} \frac{a_n}{w^n} dw = \sum_{n=2}^{\infty} \frac{a_n}{n-1} \frac{1}{z^{n-1}}$$

v) Differentiation

It is not true in general that if

$f(z) \sim \sum_{n=0}^{\infty} \frac{a_n}{z^n}$ then $f'(z) \sim \sum_{n=0}^{\infty} \frac{d}{dz} \frac{a_n}{z^n}$

e.g. $f(x) = e^{-x} \sin(e^x) \qquad x \geq 0$

$f(x)$ has no asymptotic expansion (all coefficients are zero).

$f'(x) = \sin(e^x) - e^{-x} \sin(e^x)$.

The term $\sin(e^x)$ does not tend to a limit as $x \to \infty$ and the coefficients do not exist.

Example

$$F(z) = \int_0^{\infty} \frac{e^{-t}}{t+z} dt$$

$$\frac{1}{t+z} = \frac{1}{z} + \frac{(-t)}{z^2} + \cdots + \frac{(-t)^{n-1}}{z^n} + \frac{(-t)^n}{(z+t)z^n}$$

$$F(z) = \sum_{r=0}^{n-1} \int_0^{\infty} \frac{e^{-t}(-1)^r t^r}{z^{r+1}} dt + R_n(z)$$

where $R_n(z) = \int_0^{\infty} \frac{e^{-t}(-t)^n}{z^n(z+t)} dt$

$|z^n R_n z| \leq \int_0^{\infty} \frac{e^{-t} t^n}{|z+t|} dt \leq \int_0^{\infty} \frac{e^{-t} t^n}{|z|} dt \ \to 0$ as $|z| \to \infty$ if $R(z) > 0$

If $z = x$, real and positive

$R_n(x) = (-1)^n \int_0^\infty \frac{t^n e^{-t}}{x^n(x+t)} dt$

$|R_n(x)| \leq \int_0^\infty \frac{t^n e^{-t}}{x^{n+1}} = \frac{n!}{x^{n+1}}$

$|R_n(x)| \leq |\text{next term in the series}|$

The series $\sum_{r=0}^{n-1} \frac{(-1)^r r!}{z^{r+1}}$ diverges (by the ratio test).

Euler-Maclaurin Formula

$$\int_0^{mh} F(x)dx = h\left[\frac{1}{2}F(0) + F(1) + \cdots + F(m-1) + \frac{1}{2}F(m)\right]$$
$$-\frac{h^2}{2!}B_1[F'(mh) - F'(0)] + \frac{h^4 B_2}{4!}[F'''(mh) - F'(0)]$$
$$+(-1)^n \frac{h^{2n} B_n}{(2n)!}[F^{(2n-1)}(mh) - F^{(2n-1)}(0)] + R_n$$

$$R_n = \frac{h^{2n+3}}{(2n+2)!} \int_0^{mh} \Phi_{2n+2}(t) F^{(2n+2)}(ht)dt$$

Where $\Phi_n(t) = \phi_n(t) \qquad 0 \leq t \leq 1 \qquad$ (Bernoulli polynomial)

$\Phi_n(t+1) = \Phi_n(t)$