Exam Question
Topic: Tangent Plane
Write down the formula for the directional derivative in the direction θ at a point P on the surface given by the equation $z = f(x, y)$.
Prove that the direction of maximum slope and the direction of zero slope are always at right angles.
Find the equation of the tangent plane at the point $(1, 2, -5)$ to the surface given by the equation
$$z = xy^3 - 3(x + y).$$

Solution
The directional derivative is given by
$$D(\theta) = f_x \cos\theta + f_y \sin\theta,$$
where f_x and f_y are evaluated at P.
The direction of zero slope is given by $\tan\theta = -f_x/f_y$.
Now $D'(\theta) = f_x \sin\theta + f_y \cos\theta$ and so $D'(\theta) = 0$ when $\tan\theta = f_y/f_x$.
The product of these two directions is -1 and so they are at right-angles.

$$\begin{aligned} z &= xy^2 - 3(x + y) \\ z_x &= y^2 - 3 = 2^2 - 3 = 1 \text{ at } P \\ z_y &= 2xy - 3 = 2.1.2 - 3 = 1 \text{ at } P \end{aligned}$$

The equation of the tangent plane is therefore
$$z + 5 = 1(x - 1) + 1(y - 2); \quad x + y - z = 8.$$