Question
Given that

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and B is such that $AB = BA$, show that B must have the form

$$\begin{pmatrix} a & b & c \\ b & a+c & b \\ c & b & a \end{pmatrix}$$

where a, b and c are arbitrary. (HINT: Let

$$B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & g & i \end{pmatrix}$$

and calculate both AB and BA. Compaire the entries in these two matricies to show that $d = f = h = b$ etc...)

Answer

$$AB = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} d & e & f \\ a+g & b+h & c+i \\ d & e & f \end{pmatrix}$$

$$BA = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} b & a+c & b \\ e & d+f & e \\ h & g+i & h \end{pmatrix}$$

and so $d = h = f = b$.
Also: $e = a + c = g + i = a + g = c + i$ when

$$a + c = c + i \implies a = i$$
$$a + c = a + g \implies c = g$$

Therefore, d, e, f, g, h and i are uniquely determined by a choice of a, b and c. Hence B must have the form:

$$B = \begin{pmatrix} a & b & c \\ b & a+c & b \\ c & b & a \end{pmatrix}.$$