Question

An anti-clockwise rotation of the plane (centre the origin) through an angle α corresponds to the matrix

$$C(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix},$$

and the reflection of the plane in the line through the origin making an angle β with the x-axis (where β is measured anti-clockwise) corresponds to the matrix

$$S(\beta) = \begin{pmatrix} \cos(2\beta) & \sin(2\beta) \\ \sin(2\beta) & -\cos(2\beta) \end{pmatrix}.$$

Show that

(i) $C(\alpha)C(\gamma) = C(\alpha+\gamma)$ (a rotation followed by a rotation gives a rotation);

(ii) $C(\alpha)S(\beta) = S(\frac{2\beta+\alpha}{2})$ (a reflection followed be a rotation gives a reflection);

(iii) $S(\beta)C(\alpha) = S(\frac{2\beta-\alpha}{2})$ (a rotation followed by a rotation gives a reflection)

(iv) $S(\beta)S(\gamma) = C(2\beta - 2\gamma)$ (a reflection followed by a reflection gives a rotation)

(Hint: You will need the following expansions:

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$$

$$\cos(A \pm B) = \cos(a)\cos(B) \mp \sin(A)\sin(B)\ )$$

Answer

(i) $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix}$

$$\begin{aligned} &= \begin{pmatrix} \cos\alpha\cos\gamma - \sin\alpha\sin\gamma & -[\sin\alpha\cos\gamma + \cos\alpha\sin\gamma] \\ \sin\alpha\cos\gamma + \cos\alpha\sin\gamma & \cos\alpha\cos\gamma - \sin\alpha\sin\gamma \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix} \\ &= C(\alpha+\gamma) \end{aligned}$$

(ii) $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{pmatrix}$

$$\begin{aligned} &= \begin{pmatrix} \cos 2\beta\cos\alpha - \sin 2\beta\sin\alpha & \sin 2\beta\cos\alpha + \cos 2\beta\sin\alpha \\ \sin 2\beta\cos\alpha + \cos 2\beta\sin\alpha & -[\cos 2\beta\cos\alpha - \sin 2\beta\sin\alpha] \end{pmatrix} \\ &= \begin{pmatrix} \cos(2\beta+\alpha) & \sin(2\beta+\alpha) \\ \sin(2\beta+\alpha) & -\cos(2\beta+\alpha) \end{pmatrix} \\ &= S\left(\frac{2\beta+\alpha}{2}\right) \end{aligned}$$

(iii) $\begin{pmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$

$$\begin{aligned} &= \begin{pmatrix} \cos 2\beta\cos\alpha + \sin 2\beta\sin\alpha & \sin 2\beta\cos\alpha - \cos 2\beta\sin\alpha \\ \sin 2\beta\cos\alpha - \cos 2\beta\sin\alpha & -[\cos 2\beta\cos\alpha + \sin 2\beta\sin\alpha] \end{pmatrix} \\ &= \begin{pmatrix} \cos(2\beta-\alpha) & \sin(2\beta-\alpha) \\ \sin(2\beta-\alpha) & -\cos(2\beta-\alpha) \end{pmatrix} \\ &= S\left(\frac{2\beta-\alpha}{2}\right) \end{aligned}$$

(iv) $\begin{pmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{pmatrix} \begin{pmatrix} \cos 2\gamma & \sin 2\gamma \\ \sin 2\gamma & -\cos 2\gamma \end{pmatrix}$

$$\begin{aligned} &= \begin{pmatrix} \cos 2\beta\cos 2\gamma + \sin 2\beta\sin 2\gamma & -[\sin 2\beta\cos 2\gamma - \cos 2\beta\sin 2\gamma] \\ \sin 2\beta\cos 2\gamma - \cos 2\beta\sin 2\gamma & \cos 2\beta\cos 2\gamma + \sin 2\beta\sin 2\gamma \end{pmatrix} \\ &= \begin{pmatrix} \cos(2\beta-2\gamma) & -\sin(2\beta-2\gamma) \\ \sin(2\beta-2\gamma) & \cos(2\beta-2\gamma) \end{pmatrix} \\ &= C(2\beta-2\gamma) \end{aligned}$$