Question

A crystal lattice is generated by the vectors $\mathbf{a}_1 = 3\mathbf{i}$, $\mathbf{a}_2 = \mathbf{i} + 2\mathbf{j}$, $\mathbf{a}_3 = \mathbf{i}+\mathbf{j}+\mathbf{k}$, based at the origin. A general point $\mathbf{x}$ of the lattice can be expressed as

$$\mathbf{x} = r\mathbf{a}_1 + s\mathbf{a}_2 + t\mathbf{a}_3$$

where r, s and t are scalars. Write down the matrix which allows one to convert these "$\mathbf{a}_i$-coordinates" into standard "$\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ coordinates", and use it to convert the following vectors into standard form:

(i) $\mathbf{u} = 3\mathbf{a}_1 + 2\mathbf{a}_2 + 4\mathbf{a}_3$

(ii) $\mathbf{v} = 2\mathbf{a}_1 - 4\mathbf{a}_2 + 5\mathbf{a}_3$

(iii) $\mathbf{w} = -3\mathbf{a}_1 - 2\mathbf{a}_2 + 4\mathbf{a}_3$

Do the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ lie on a common plane through the origin?

Answer

The matrix is $\begin{pmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, so that a point of the lattice with $\mathbf{a}_i$-coordinates (r, s, t) has $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$-coordinates (x, y, z) where:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r \\ s \\ t \end{pmatrix}$$

(i)

$$\mathbf{u} = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ -4 \end{pmatrix} \text{ so } \mathbf{u} = 7\mathbf{i} - 4\mathbf{k}$$

(ii)

$$\mathbf{v} = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -4 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \\ 5 \end{pmatrix} \text{ so } \mathbf{v} = 7\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$$

(iii)

$$\mathbf{w} = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -7 \\ 0 \\ 4 \end{pmatrix} \text{ so } \mathbf{w} = -7\mathbf{i} + 4\mathbf{k}$$

Note that $\mathbf{u} = -\mathbf{w}$ so that $\mathbf{u}$ and $\mathbf{w}$ are parallel vectors. Hence $\mathbf{u}$ and $\mathbf{w}$ lie on a common line through the origin, and so $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ lie on a common plane through the origin.