Question

Define the logarithm of a non-zero complex number and discuss its multi-valued nature. If $a, b \in \mathbf{C}$ and $a \neq 0$ define a^b. Show that a^i has infinitely many values.

Show how z^i changes its value by choosing a particular value w_0 of z_0^i and then passing once around the circle with centre o and radius $|z_0|$. Explain how you would define a single-valued branch $f(z)$ of z^i which gives an analytic function in the upper half plane U. If $r = |z|$, $\theta = \arg z$ calculate for the particular branch you have chosen, $|z^i|$ and $\arg z^i$.

Find the image $f(u)$ of the upper half plane under f. Also, find the images of the following subsets of U.

i) $\{z \in U | \, |z| = r_1\}$

ii) $\{z \in U | \arg z = \theta_1\}$.

Answer

If $z = e^w$ then we say that w is a logarithm of z. Now e^w is periodic with period $2\pi i$, so $z = e^{w+2n\pi i}$, and $w \neq 2n\pi i$ is also a logarithm of z, for $n \in \mathbf{Z}$.

Now let $w = x + iy$ so $z = e^x e^{iy}$ thus $e^x = |z|$ and $y = \arg z$.

So $w = \ln |z| + i \arg z$, and again the multi-valued nature is clear from that of $\arg z$.

We define a^b by $a^b = \exp(b \log a)$

So $a^i = \exp(i \log a) = \exp(i[\log |a| + i(\text{Arg}a + 2n\pi)])$

$= \exp i \ln |a| \exp(-\text{Arg}a) \exp(-2n\pi) \quad n \in \mathbf{Z}$

The first two factors are uniquely defined and non-zero. The third factor is different for each $n \in \mathbf{Z}$.

Let z pass round the circle centre O, radius $|z_0|$. Choosing the value of z^i with $n = 0$ we have

$z^i = \exp(i \ln |z_0|) \exp(-\text{Arg}z) = K \exp(-\text{Arg}z)$

so with the principal value of Arg$z$ satisfying $-\pi \leq \text{Arg}z < \pi$, z^i changes as follows:

when $z = -|z_0|$, i.e. $\text{Arg}z = -\pi$, $z^i = Ke^{\pi}$ as z moves round the circle Arg$z$ increases and so z^i assumes values of the form $Ke^{-\pi t} \quad -1 \leq t < 1$.

As $\text{Arg}z \to \pi$, $\quad z^i \to Ke^{-\pi}$, so there is a discontinuity at $-|z_0|$ of magnitude $K(e^{\pi} - e^{-\pi})$. To choose a single-valued branch analytic for im$z > 0$ the above will do.

So if $r = |z|$ and $\theta = \text{Arg} z$ then $z^i = \exp(i \log r) \exp(-\theta)$
So $|z^i| = e^-\theta, \quad \arg(z^i) = \log r$
If U is the upper half plane, $0 < \theta < \pi$ and $r > 0$, so $\log r$ takes all real values and $e^{-\theta}$ takes values between 1 and $e^{-\pi}$. So the image of U is the open annulus $e^{-\pi} < |w| < 1$.

i) $\{z \in U | \, |z| = r_1\}$ - the subset of this annulus satisfying $\arg w = \log r_1$. i.e. a line segment.

ii) $\{z \in U | \arg z = \theta_1\}$. - a circle of radius $e^{-\theta}$.

DIAGRAM