Question

i) Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^2}$ by contour integration.

ii) Prove that if R is a positive real number then

$$\int_R^{R+iR} \frac{e^{iz}dz}{z}, \qquad \int_{-R+iR}^{R+iR} \frac{e^{iz}dz}{z} \qquad \text{and} \qquad \int_{-R}^{-R+iR} \frac{e^{iz}dz}{z}$$

all tend to 0 as $R \to \infty$, where in each case the integral is taken over a straight line. Hence prove that

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

Answer

i) Let $f(z) = \dfrac{1}{(z^2+4)^2}$. This is analytic except for poles of order 2 at $z = \pm 2i$.

For all $x \quad \dfrac{1}{(x^2+4)^2} \le \dfrac{1}{x^4}$ and $\displaystyle\int^{\infty} \frac{1}{x^4}$ converges.

So by comparison $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+4)} dx$ converges.

We integrate $f(z)$ round the contour Γ.

DIAGRAM

For $R > 2$ there is a pole of order 2 at $z = 2i$ inside Γ.

$$\text{res}(f, 2i) = \frac{d}{dz}(z-2i)^2 f(z)\Big|_{z=2i}$$

$$= \frac{d}{dz}\frac{1}{(z+2i)^2} = \frac{-2}{(z+2i)^3}\Big|_{z=2i} = \frac{-i}{32}$$

$$\int_\Gamma f(z)dz = 2\pi i \frac{(-1)}{32} = \frac{\pi}{16}$$

On $C_2 \quad |f(z)| = \dfrac{1}{|z^2+4|^2} \le \dfrac{1}{(|z|^2-4)^2} = \dfrac{1}{(R^2-4)^2}$ for $R \ge 2$.

So $\left|\int_{C_2} f(z)dz\right| \leq \dfrac{\pi R}{R^2-4} \to 0$ as $R \to \infty$

$\int_\Gamma = \int_{-R}^{R} + \int_{C_2}$ so letting $R \to \infty$ gives $\int_{-\infty}^{\infty} \dfrac{dx}{(x^2+4)^2} = \dfrac{\pi}{16}$

ii) DIAGRAM

On C_1 $\quad \left|\dfrac{e^{iz}}{z}\right| = \dfrac{|e^{i(R+it)}|}{|z|} \leq \dfrac{e^{-t}}{R}$

So $\left|\int_{C_1} \dfrac{e^{iz}}{z} dz\right| \leq \int_0^R \dfrac{e^{-t}}{R} dt = \dfrac{1-e^{-R}}{R} \to 0$ as $R \to \infty$

On C_2 $\quad \left|\dfrac{e^{iz}}{z}\right| = \dfrac{|e^{i(t+iR)}|}{|z|} = \dfrac{e^{-R}}{|z|} \leq \dfrac{e^{-R}}{R}$

So $\left|\int_{C_2} \dfrac{e^{iz}}{z} dz\right| \leq \dfrac{e^{-R}}{R} 2R \to 0$ as $R \to \infty$

On C_3 $\quad \left|\dfrac{e^{iz}}{z}\right| = \dfrac{|e^{i(-R+it)}|}{|z|} \leq \dfrac{e^{-t}}{R}$

and again $\int_{C_3} \to 0$ as $R \to \infty$, as with $\int_{C_1}$

Now $z\dfrac{e^{iz}}{z} = e^{iz} \to 1$ as $z \to 0$ so $\dfrac{e^{iz}}{z}$ has a simple pole at $z = 0$ with residue 1.

So $\dfrac{e^{iz}}{z} = \dfrac{1}{z} + g(z)$ where $g(z)$ is analytic near 0.

So $\exists K,\ M$ such that $|g(z)| \leq M$ for $|z| \leq K$.

Thus for the small semi-circle C, $z = -re^{-it} \quad 0 \leq t \leq \pi \quad r \leq K$

$\int_C \dfrac{e^{iz}}{z} = \int_C \dfrac{1}{z} dz + \int_C g(z)dz$

Now $\left|\int_C g(z)dz\right| \leq M\pi r \to 0$ as $r \to 0$

and $\int_C \dfrac{1}{z} dz = \int_0^{\pi} \dfrac{ire^{-it}}{-re^{-it}} = -\pi i$

Inside Γ, $\quad \dfrac{e^{iz}}{z}$ is analytic. Hence $\int_\Gamma \dfrac{e^{it}}{z} \to 0$.

So letting $R \to \infty$, $r \to 0$ gives

$$\int_{-\infty}^{0} \frac{e^{ix}}{x} dx + \int_{0}^{\infty} \frac{e^{ix}}{x} dx - \pi i = 0$$

So $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x} = \pi$ i.e. $\displaystyle\int_{0}^{\infty} \frac{\sin x}{x} = \frac{\pi}{2}$