Question

i) Determine the radius of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n}$.

Find one point on the circle of convergence where the series diverges, and two distinct points where the series converges.

ii) Express $\dfrac{z+1}{z-1}$ as a Taylor series centred at the origin. What is the largest region in which this series converges to the function?

Express the same function as a Laurent series in $|z| > 1$.

iii) Find all the singular points of the function

$$\frac{\left(z - \frac{\pi}{2}\right)}{(e^z - 1)^3 \cos z}$$

and determine their natures.

Answer

i) Let $u_n = \dfrac{z^n}{n}$ $\qquad \left|\dfrac{u_{n+1}}{u_n}\right| = \dfrac{n}{n+1}|z| \to |z|$ as $n \to \infty$ therefore $R = 1$.

The series diverges at $z = 1$ since $\sum \frac{1}{n}$ diverges.

At $z = -1$ the series is $\sum \frac{(-1)^n}{n}$ which is convergent by the Leibniz test.

At $z = i$ the series is $\frac{i}{1} - \frac{1}{2} - \frac{i}{3} + \frac{1}{4} + \frac{i}{5} - \frac{1}{6} - \cdots$

$= -\frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \cdots + i(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots)$

so both real and imaginary parts converge by the Leibniz test.

ii) $\dfrac{1}{z-1} = -(1 + z + z^2 + \cdots)$

so $\dfrac{z+1}{z-1} = -(1+z)(1 + z + z^2 + \cdots) = -(1 + 2z + 2z^2 + 2z^3 + \cdots)$

which converges for $|z| < 1$.

For $|z| > 1$, $\dfrac{z+1}{z-1} = \dfrac{z+1}{z\left(1-\frac{1}{2}\right)}$

$= \left(1+\dfrac{1}{z}\right)\left(1+\dfrac{1}{z}+\dfrac{1}{z^2}+\cdots\right) = 1+\dfrac{2}{z}+\dfrac{2}{z^2}+\dfrac{2}{z^3}+\cdots$

iii) singularities occur at places where $e^z = 1$ and $\cos z = 0$

i.e. $z = 2n\pi i$ and $z = (2n+1)\frac{\pi}{2}$

Now $\dfrac{z}{e^z-1} = \dfrac{1}{1+\frac{z}{2!}+\cdots} \to 1$ as $z \to 0$

So $z^3 f(z) \to \dfrac{-\frac{\pi}{2}}{1.1} \neq 0$ as $z \to 0$

Thus $f(Z)$ has a pole of order 3 at $z = 0$, and by periodicity of e^z at $z = 2n\pi i$.

Letting $p(z) = \dfrac{z-(2n+1)\frac{\pi}{2}}{\cos z}$

Use L'Hopital's rule

$p(z) \to \lim_{z\to(2n+1)\frac{\pi}{2}} \dfrac{1}{\sin z} \neq 0$ as $z \to (2n+1)\frac{\pi}{2}$

so $f(z)$ has a removable singularity at $z = \frac{\pi}{2}$, and simple poles at $z = (2n+1)\frac{\pi}{2}$ $\quad n \in \mathbf{Z} \quad n \neq 1$